Math3C: Final

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Note that these answers are not reviewed by Dr. Weisbert and/or may include some errors. If you find one or more, let me know and I'll edit the document. My email is wesleytk@ucla.edu.

The average score for Lecture 3 was 68% with a standard deviation of 19%. The median was 71%, whereas the max was 100% and min was 13%.

C Level Questions

C1. You can have either eggs, pancakes, or cereal for breakfast. You can have only one type of breakfast per day and you must have breakfast each day. How many different ways can you plan breakfast for a week? There are seven days in a week.

You have 7 days, and there are 3 choices for breakfast each day, giving:

$$3^7$$

C2. How many ways can you order the letters *aaabbccccd*?

This word is 10 letters, but there are 3 a's, 2 b's, 4 c's and 1 d. This is just a permutation problem:

$$\frac{10!}{3!2!4!1!}$$

C3. You are to bring five sodas to a party in a cooler. There are 12 types of sodas to choose from: 7up, Pepsi, Mountain Dew, Fanta, Orangina, Root Beer, Dr. Pepper, Sprite, Coke, Fresca, Diet Coke, and Ginger Ale. How many different collections of five sodas can you bring?

This is a classic rings and fingers problem, where there are 12 fingers (types of soda) and 5 identical rings (sodas to bring). This gives us:

$$\binom{n+k-1}{n} = \binom{n+k-1}{k-1} = \binom{12+5-1}{5} = \binom{12+5-1}{12-1}$$

C4. A bag contains five red balls and two green balls. You remove exactly two balls from the bag. What is the probability that both balls are red?

Remember it's:

$$P = \frac{\#win}{\#total}$$

To get the number of different ways to get 2 red balls (winning), you have to select 2 of the 5, which gives you: $\binom{5}{2}$. To get the number of different ways you can get 2 balls, you need to select 2 of the 5+2, or, $\binom{5+2}{2}$. That gives us:

$$P = \frac{\binom{5}{2}}{\binom{10}{2}}$$

C5. You have a fair three sided die, a fair four sided die, and a fair six sided die. You first toss the three sided die. If it lands on 1 or 2, you roll the four sided die and record the number on which it lands. If it lands on 3, you roll the six sided die and record the number on which it lands. What is the probability that you record a 4?

This is just a law of total probability problem:

$$P(4) = P(4|1,2)P(1,2) + P(4|3)P(3) = \frac{1}{4}\frac{2}{3} + \frac{1}{6}\frac{1}{3}$$

C6. A disease has a prevalence of $\frac{1}{5}$ in a population. A test for the disease gives 10% false positives and 5% false negatives. What is the probability that you have the disease, given that you test negative?

You guys did this problem over and over again this quarter. Let's find the meaningful probabilities:

$$\begin{aligned}
P(D) =& \frac{1}{5} \\
P(D^c) =& 1 - \frac{1}{5} = \frac{4}{5} \\
P(-|D) =& \text{false negatives} = 0.05 \\
P(+|D) =& 1 - P(-|D) = 0.95 \\
P(+|D^c) =& \text{false positives} = 0.1 \\
P(-|D^c) =& 1 - P(+|D^c) = 0.9
\end{aligned}$$

Then we have memorized Bayes formula:

$$P(D|-) = \frac{P(-|D)P(D)}{P(-|D)P(D) + P(-|D^c)P(D^c)} = \frac{0.05\frac{1}{5}}{0.05\frac{1}{5} + 0.9\frac{1}{5}}$$

C7. You roll a fair six sided die 10 times. The sides are labeled 1,2,3,4,5, and 6. What is the probability that you roll a 1 exactly three times?

Each roll is independent, so the probabilities multiply, giving us:

$$P(111) = P(1)^3 = \frac{1}{6^3}$$

C8. You toss a fair coin 20,000 times. Use Chebychev's inquality to estimate the probability that you do not land on heads between 45% and 55% of the time.

Note that this is the easy way to do Chebychev's, NOT the hard way. Chebychev's says:

$$P(|X - E(X)| \geq c) \leq \frac{Var(X)}{c^2}$$

In this case, X is the probability of heads. It's a fair coin, so $E(X) = 50\%$. The difference between the mean and those bounds are $c = 5\% = 0.05$. The other trick we know is the law of weak numbers. Plugging that in:

$$P(|X - E(X)| \geq 0.05) \leq \frac{p(1-p)}{n(0.05)^2} = \frac{1}{4 \cdot 20,000 \cdot (0.05)^2}$$

C9. Your partner has herpes and you do not. The probability that you contract herpes from your partner is $\frac{1}{10}$ each time you are exposed. What is the probability that you contact herpes from your partner on the fifth exposure and not before?

This is a way to say given. Remember:

$$P(herpes = 5 | herpes \geq 5) = \frac{P(herpes = 5 \cap herpes \geq 5)}{P(herpes \geq 5)} = \frac{P(herpes = 5)}{P(herpes \geq 5)}$$

Now we recognize that this is a Geometrically distributed variable:

$$P(herpes = 5 | herpes \geq 5) = \frac{\left(\frac{9}{10}\right)^4 \frac{1}{10}}{\left(\frac{9}{10}\right)^4} = \frac{1}{10}$$

C10. You roll a fair six sided die 30 times. What is the expected number of times the die lands on 2?

This is a mean of a Binomially distributed variable. The probability of landing on 2 on each individual trial, p, is $\frac{1}{6}$. The number of trials, n is 30. Then we know that:

$$E(X) = np = 30\frac{1}{6} = 5$$

C11. Suppose that a continuous random variable X has a probability density function of ρ where

$$\rho(x) = \begin{cases} cx & \text{if } 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

What must c be equal to?

We remember that probability densities must integrate to 1. That gives us:

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} \rho(x)dx \\ 1 &= \int_0^2 cxdx \\ 1 &= \frac{cx^2}{2}\Big|_{x=0}^{x=2} \\ 1 &= c\left[\frac{2^2}{2} - \frac{0^2}{2}\right] \\ 1 &= c \cdot 2 \\ \frac{1}{2} &= c \end{aligned}$$

Given the homework, we also notice that the probability density is always positive, so it still works.

C12. The half-life of an atom is 10 days. What is the probability that the atom decays within 25 days?

We memorized this about exponential distributions:

$$\begin{aligned} P(T \leq t) =& 1 - e^{-\lambda t} \\ P(T \leq 25) =& 1 - \left(\frac{1}{2}\right)^{\frac{25}{10}} \end{aligned}$$

B Level Questions

B1. Seven people stand in line for a photograph. Three of the people are good friends and insist on standing together (they look different however). How many photographs are possible?

There are 2 different ways to do this, which result in equivalent answers. First, treat all of the friends as one person. Ordering those 5 objects, gives us 5! options. Then we must worry about the order of the friends, which gives us an additional 3!. For every order of 5 objects, you have every order of the friends, so you multiply those two to get:

$$5!3!$$

B2. Your friend has five shirts, three skirts and eight hats. She also has three identical gold rings. An outfit consists one shirt, one skirt, one hat, and either two or three rings. Outfits are considered different if the pattern of the rings on the 10 fingers is different. How many outfits can be made?

There is an easy part to this and a harder part that people messed up. For the easier part, you can only wear one of the clothing items, and have a few different options. So, ignoring the rings, we have:

$$5 \cdot 3 \cdot 8$$

Then we have to worry about the rings and fingers. You have 10 fingers and either 2 or 3 rings. Either means that you add those two possibilities. For the number of ring combinations, you need:

$$\binom{10+2-1}{2} + \binom{10+3-1}{3}$$

To get the final answer, you multiply:

$$5 \cdot 3 \cdot 8 \cdot \binom{10+2-1}{2} + \binom{10+3-1}{3}$$

B3. You have a class of 30 students. The students indend to form five clubs: two math clubs, a history club, a physics club and a poetry club. The math clubs will have four members each and members can only be in one of the two clubs. The two math clubs cannot be distinguished, that is, members do not know which math club they are in since the clubs are not named. The history, physics, and poetry clubs ahve two, three, and four members respectively. Members can serve multiple times between the clubs, expect in the math clubs. So, Joe can be in poetry, history and math club simultaneously but not in both math clubs. In how many ways can the students be separated into clubs?

Let's do the less complicated clubs first, that are all labelled (history, physics and poetry). Note that everyone can be in each of these clubs, giving us:

$$\binom{30}{2}\binom{30}{3}\binom{30}{4}$$

Now let's talk about the math clubs. They are indistinguishable, and you can't be in both, giving us:

$$\frac{1}{2!}\binom{30}{4}\binom{26}{4}$$

The leading $\frac{1}{2!}$ is for the indistinguishable (deordering). The drop to 26 is because you chose 4 for the first club, so you can no longer choose them for the second club. To get the final amount, we multiply:

$$\binom{30}{2}\binom{30}{3}\binom{30}{4}\frac{1}{2!}\binom{30}{4}\binom{26}{4}$$

B4. You have 30 cards: 10 red, 10 green, and 10 blue. The cards of each color are labeled 1, 2, 3, . . ., 10. You randomly draw five cards from the 30. What is the probability that one value will be repeated three times and a second value, different from the first, will be repeated two times.

Remember probabilities are:

$$P = \frac{\#wins}{\#total}$$

To get the number total, we just select 5 cards from 30: $\binom{30}{5}$. For the number of wins, we need to choose the values, which are all distinguishable. We've got 10 choices for the first card, and 9 choices for the second card. We multiply that to get $10 \cdot 9$. Now we need to choose the colors for those cards. For the triplet, you have 3 cards to choose the color for and 3 colors, giving us: $\binom{3}{3}$. For the pair, you have 2 cards to choose the color for and 3 colors, giving us: $\binom{3}{2}$. Plugging that all in:

$$P = \frac{10 \cdot 9 \cdot \binom{3}{3}\binom{3}{2}}{\binom{30}{5}}$$

B5. You have ten red cards, ten green cards, five blue cards, and eight yellow cards. You randomly draw four cards. What is the probability that you draw a card of each color given that you draw at least one blue card?

Let's go back to givens:

$$P(\text{each color}|\geq 1B) = \frac{P(\text{each color}\cap \geq 1B)}{P(\geq 1B)} = \frac{P(\text{each color})}{P(\geq 1B)}$$

To get the probabilities, we need to think about the definition of probability again:

$$P = \frac{\#wins}{\#total}$$

For the numerator:

$$P(\text{each color}) = \frac{10 \cdot 10 \cdot 5 \cdot 8}{\binom{33}{4}}$$

We have $\binom{33}{4}$ choices for how many different sets of 4 cards you can select from 33. The numerator is the number of ways you can get a red card (10) times the ways you get a green card (10) times the ways you get a blue card (5) times the ways you get a yellow card (8).

Then the denominator, we have:

$$\begin{aligned}
P(\geq 1B) =& 1 - P(0B) = 1 - \frac{\binom{28}{4}}{\binom{33}{4}} \\
=& P(1B) + P(2B) + P(3B) + P(4B) = \frac{\binom{5}{1}\binom{28}{3} + \binom{5}{2}\binom{28}{2} + \binom{5}{3}\binom{28}{1} + \binom{5}{4}}{\binom{33}{4}}
\end{aligned}$$

Now dividing those, we get the answer:

$$\frac{\frac{10\cdot 10\cdot 5\cdot 8}{\binom{33}{4}}}{1-\frac{\binom{28}{4}}{\binom{33}{4}}} = \frac{\frac{10\cdot 10\cdot 5\cdot 8}{\binom{33}{4}}}{\frac{\binom{5}{1}\binom{28}{3}+\binom{5}{2}\binom{28}{2}+\binom{5}{3}\binom{28}{1}+\binom{5}{4}}{\binom{33}{4}}} = \frac{10\cdot 10\cdot 5\cdot 8}{\binom{33}{4}-\binom{28}{4}} = \frac{10\cdot 10\cdot 5\cdot 8}{\binom{5}{1}\binom{28}{3} + \binom{5}{2}\binom{28}{2} + \binom{5}{3}\binom{28}{1} + \binom{5}{4}}$$

B6. A disease has a prevalence of $\frac{1}{7}$ in a population. Test A gives 10% false positives and 2% false negatives. Test B gives 3% false positives and 12% false negatives. The false positive and false negative rates for teh test depend only on whether or not the individual tested has the disease. What is the probability that you have the disease if you test positive on Test A and negative on Test B?

The derivation of this formula is in my homework solutions.

$$P(D| +_A -_B) = \frac{P(+_A|D)P(-_B|D)P(D)}{P(+_A|D)P(-_B|D)P(D) + P(+_A|D^c)P(-_B|D^c)P(D^c)}$$

Let's figure out each of the possibilities:

$$\begin{aligned}
P(D) =& \frac{1}{7} \\
P(D^c) =& 1 - P(D) = \frac{6}{7} \\
P(+_A|D^c) =& \text{false positives} = 0.1 \\
P(-_A|D^c) =& 1 - P(+_A|D^c) = 0.9 \\
P(-_A|D) =& \text{false negatives} = 0.02 \\
P(+_A|D) =& 1 - P(-_A|D) = 0.98 \\
P(+_B|D^c) =& \text{false positives} = 0.03 \\
P(-_B|D^c) =& 1 - P(+_A|D^c) = 0.97 \\
P(-_B|D) =& \text{false negatives} = 0.12 \\
P(+_B|D) =& 1 - P(-_A|D) = 0.88
\end{aligned}$$

Plugging those in gives us:

$$P(D| +_A -_B) = \frac{\frac{1}{7}\cdot 0.98 \cdot 0.12}{\frac{1}{7}\cdot 0.98\cdot 0.12 + \frac{6}{7}\cdot 0.97 \cdot 0.1}$$

B7. Your partner has a disease. The probability that you contract the disease on a single exposure is $\frac{1}{20}$. You are exposed to your partner eight times. What is the probability that you contract the disease?

This is just a 1- trick:

$$P(\text{contract}) = P(\text{contract} \geq 1) = 1 - P(\text{contract} = 0) = 1 - P(\text{don't contract}) = 1 - \left(\frac{19}{20}\right)^8$$

The probability that you don't contract on any one trial is $\frac{19}{20} = 1 - \frac{1}{20}$. You need to do that 8 times in a row. When you do things in a row, you multiply.

B8. You toss a coin 100,000 times and find that 60% of the time the coin has landed on heads. You determien the probability for the coin to land on heads is between 58% and 62%. Find a reasonable lower bound on the probability that your determination is correct.

Your Chebychev bells should be going off repeatedly right now. First, our mean, $E(X) = 0.60 = 60\%$, and our $c = 0.02 = 2\%$ because the bounds are 2% on either side of that mean. For some reason, some people used 4%, which is not correct. Then, we have:

$$\begin{aligned}
P(|X - E(X)| \geq c) \leq& \frac{Var(X)}{c^2} \\
P(|X - E(X)| \geq c) =& 1 - P(|X - E(X)| \leq c) \\
1 - P(|X - E(X)| \leq c) \leq& \frac{Var(X)}{c^2} \\
P(|X - E(X)| \leq c) \geq& 1 - \frac{Var(X)}{c^2} \\
& \text{Weak law of large numbers for Binomial proportions} \\
P(|X - E(X)| \leq c) \geq& 1 - \frac{1}{4nc^2} \\
\geq& 1 - \frac{1}{4 \cdot 100,000 \cdot 0.02^2}
\end{aligned}$$

Where n is the number of trials, in this case 100,000.

B9. You have a fair six sided die and a bag with 10 balls: four red balls and six green balls. You roll the die. If it lands on 1, you randomly pick one ball from the bag. If it does not land on 1, you roll again. If it lands on 1 on the second try, you randomly pick two balls from the bag. If it does nto land on 1 on the second try, you roll again. If it lands on 1 on the third try, you randomly pick three balls from the bag. If it does not land on 1 on the third try, you randomly pick four balls from the bag. What is the probability that you choose at least 1 red ball?

This one is a law of total probability problem with an at least trick. Not a lot of people tried this one, and of the people that tried, not a lot got full credit. Let's remember the law of total probability in this context, where X is a number other than 1 when rolling:

$$P(\geq 1R) = P(\geq 1R|1)P(1) + P(\geq 1R|X1)P(X1) + P(\geq 1R|1)P(XX1) + P(\geq 1R|XXX)P(XXX)$$

Let's plug in the probability of each roll combination:

$$P(\geq 1R) = P(\geq 1R|1)\frac{1}{6} + P(\geq 1R|X1)\frac{5}{6}\frac{1}{6} + P(\geq 1R|1)\frac{5}{6}\frac{5}{6}\frac{1}{6} + P(\geq 1R|XXX)\frac{5}{6}\frac{5}{6}\frac{5}{6}$$

Now we need to think of the at least in each circumstance. First, we know that:

$$P(\geq 1R) = 1 - P(\geq 0R) = 1 - P(\text{all } G)$$

So let's switch all of that above to:

$$P(\geq 1R) = 1 - P(\text{all } G|1)\frac{1}{6} - P(\text{all } G|X1)\frac{5}{6}\frac{1}{6} - P(\text{all } G|1)\frac{5}{6}\frac{5}{6}\frac{1}{6} - P(\text{all } G|XXX)\frac{5}{6}\frac{5}{6}\frac{5}{6}$$

Now let's think about the probability of getting all green, when we have k trials. Note that there isn't replacement, so it's not Binomial (no one made that mistake). It's:

$$\begin{aligned} P(\text{all } G) =& \frac{\binom{6}{k}}{\binom{10}{k}} \\ =& \frac{6}{10}\frac{5}{9}\frac{4}{8}\frac{3}{7} \text{ if } k = 4 \end{aligned}$$

Plugging that in, we get:

$$P(\geq 1R) = 1 - \frac{6}{10}\frac{1}{6} - \frac{\binom{6}{2}}{\binom{10}{2}}\frac{5}{6}\frac{1}{6} - \frac{\binom{6}{3}}{\binom{10}{3}}\frac{5}{6}\frac{5}{6}\frac{1}{6} - \frac{\binom{6}{4}}{\binom{10}{4}}\frac{5}{6}\frac{5}{6}\frac{5}{6}$$

You also could do this without the 1- trick, but it gets really long.

B10. A coin has a probability of landing on heads equal to $\frac{1}{4}$. What is the expected number of trials for the coin to land on heads on two consecutive rolls?

This is conditional expectation straight up. You should have expected this question, and have been able to do it in your sleep. In this notation, $E(T_{HH})$ is the expected number of tosses till two heads, and the HT sequences are the outcomes of the first rolls. You need to worry about when you can start over from the beginning, then think about how many tosses you lost till you started over.

$$\begin{aligned} E(T_{HH}) =& E(T_{HH}|HH)P(HH) + E(T_{HH}|HT)P(HT) + E(T_{HH}|T)P(T) \\ =& 2\frac{1}{4^2} + [2 + E(T_{HH})]\frac{3}{4^2} + [1 + E(T_{HH})]\frac{3}{4} \\ \left[1 - \frac{3}{4^2} - \frac{3}{4}\right] E(T_{HH}) =& \frac{2 + 2 \cdot 3 + 1 \cdot 3 \cdot 4}{4^2} \\ \frac{1}{4^2}E(T_{HH}) =& \frac{20}{4^2} \\ E(T_{HH}) =& 20 \end{aligned}$$

B11. The random variables X and Y are uniformly distributed on $[0, 6]$. Suppose that X and Y are independent. If

$$Z = \max(X, Y),$$

calculate $E(Z)$.

This was another one that was a lot of work. Some people tried it, but a lot of people got confused. First, we could have memorized that if a random variable X is uniformly distributed over $[a, b]$ where $b > a$, then the distribution is:

$$\rho_X = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

In this case, that means:

$$\rho_X = \begin{cases} \frac{1}{6} & \text{if } 0 \leq x \leq 6 \\ 0 & \text{otherwise} \end{cases}$$

Otherwise, we need to solve this integral (the distribution is uniform, so it doesn't depend on x):

$$1 = \int_0^6 c dx \Rightarrow c = \frac{1}{6}$$

We also need to remember the cumulative probability of a uniform (or be able to derive it):

$$P(X \le x) = \frac{x}{6} = \int_0^x \frac{1}{6} dx$$

Okay, now we need to think about Z:

$$\begin{aligned}\rho_Z =& \rho_{X=z} P(Y \ge z) + \rho_{Y=z} P(X > z) \\ =& \frac{1}{6}\left[1 - \frac{z}{6}\right] + \frac{1}{6}\left[1 - \frac{z}{6}\right] \\ =& \frac{2}{6}\left[1 - \frac{z}{6}\right]\end{aligned}$$

Now we need to integrate that:

$$\begin{aligned}E(Z) =& \int_0^6 z\rho_z dz \\ =& \int_0^6 z\frac{2}{6}\left[1 - \frac{z}{6}\right] \\ =& \frac{2}{6}\int_0^6 zdz - \frac{2}{6^2}\int_0^6 z^2 dz \\ =& \frac{2}{6}\frac{z^2}{2}\Big|_{z=0}^{z=6} - \frac{2}{6^2}\frac{z^3}{3}\Big|_{z=0}^{z=6} \\ =& \frac{2}{6}\left[\frac{6^2}{2} - \frac{0^2}{2}\right] - \frac{2}{6^2}\left[\frac{6^3}{3} - \frac{0^3}{3}\right] \\ =& 6 - \frac{6}{3} \\ =& 6 - 2 \\ =& 4 = \frac{216}{54} = \frac{6^3}{54}\end{aligned}$$

B12. A substance has a half life of 10 days. Suppose that you have two atoms of this substance. What is the probability that both atoms will decay within 25 days?

This is a matter of remembering your exponential distribution formulae. A lot of people mixed this up. We know that:

$$P(T \le t) = 1 - \left(\frac{1}{2}\right)^{\frac{t}{t_{1/2}}}$$

That means that:

$$P(T \ge t) = \left(\frac{1}{2}\right)^{\frac{t}{t_{1/2}}}$$

We then can say that:

$$
\begin{aligned}
P(A\&B \le t) =& 1 - P(A \ge t)P(B \ge t) \\
=& 1 - \left(\frac{1}{2}\right)^{\frac{t}{t_{1/2}}} \left(\frac{1}{2}\right)^{\frac{t}{t_{1/2}}} \\
=& 1 - \left(\frac{1}{2}\right)^{\frac{2t}{t_{1/2}}} \\
=& 1 - \left(\frac{1}{2}\right)^{\frac{2\cdot 25}{10}} \\
=& 1 - \left(\frac{1}{2}\right)^{\frac{50}{10}} \\
=& 1 - \left(\frac{1}{2}\right)^{5} \\
=& 1 - \frac{1}{32} \\
=& \frac{31}{32}
\end{aligned}
$$

The most common **mistake**:

$$
\begin{aligned}
P(A\&B \le t) =& P(A \le t)P(B \le t) = \left(1 - \left(\frac{1}{2}\right)^{\frac{t}{t_{1/2}}}\right)^2 \\
=& 1 - 2\left(\frac{1}{2}\right)^{\frac{t}{t_{1/2}}} + \left(\frac{1}{2}\right)^{\frac{2t}{t_{1/2}}} \\
=& 1 - \frac{2}{2^{2.5}} + \frac{1}{32} \\
\approx& 0.98 \\
\neq& \frac{31}{32} \approx 0.97
\end{aligned}
$$

In this mistake, you don't consider the possibility that just one of the particles decayed, not both. This mistake was given 1 point partial credit of 3 points.

A Level Problems

A1. You buy hats for your ten friends for the holidays. Three of the hats are identical. When you go to the holiday dinner, it turns out that only seven of your friends have shown up. In how many ways can you distribute your presents if each person gets one hat? The presents are simultaneously distributed.

This one is a tuple problem with a twinning problem. You've seen this a lot before.

Zero case: zero of the identical hats are chosen. Then you have $\binom{7}{7}$ hats to choose, then you order them 7! ways.

First case: 1 of the identical hats are chosen. Then you have $\binom{7}{6}$ hats to choose (because you set that you already have one), then you order them in 7! ways.

Second case: 2 off the identical hats are chosen. Then you have $\binom{7}{5}$ hats to choose, then you order them in $\frac{7!}{2!}$ ways.

Third case: 3 of the identical hats are chosen. Then you have $\binom{7}{4}$ hats to choose, then you order them in $\frac{7!}{3!}$ ways.

You add across case and multiply within case, giving you:

$$\binom{7}{7}7! + \binom{7}{6}7! + \binom{7}{5}\frac{7!}{2!} + \binom{7}{4}\frac{7!}{3!}$$

If you combine the zero and first cases, you'd get:

$$\binom{8}{7}7! + \binom{7}{5}\frac{7!}{2!} + \binom{7}{4}\frac{7!}{3!}$$

The logic behind the first binomial choice is that you select 7 hats from 8 of the hats (1 of the three identical hats, and 7 of the other hats).

A2. Suppose that you have five colors and that you randomly paint the vertices of a square. What is the probability that no adjacent vertex has the same color?

This is a throw back to a harder homework problem from before. First, remember:

$$P = \frac{\#wins}{\#total}$$

Let's figure out # total. This is just having 5 options for 4 sides (5 choices for 1 side, then multiply by 5 for each additional 5), or:

$$\#total = 5^4$$

Now the harder part, # wins. Let's think of this in terms of cases. You start on one side, and have 5 options. Then you have 4 options for your next side (we'll come back to this). You then have 4 options for your third side (we'll come back to this). If your third and first side are different, then you have 3 options for your last side. If they are the same, then you have 4 options for that last side.

Let's revisit the combinations now that we know the cases of the first and third sides.

First, if the first and third sides are different. Then you've got 5 choices for the first, 4 choice for the third, 3 choices for the last, and 3 choices for the second side. That's because they can't match either the first or third. That gives us:

$$5 \cdot 4 \cdot 3 \cdot 3$$

Next, if the first and third sides are the same. Then you've got 5 choices for the first, 1 choice for the third (has to match), 4 choices for the last and 4 choices for the second. That gives us:

$$5 \cdot 1 \cdot 4 \cdot 4$$

We add those two cases together to get our final answer:

$$P = \frac{5 \cdot 4 \cdot 3^2 + 5 \cdot 4 \cdot 3^2}{5^4}$$

Note that we didn't care about the circular symmetry of the square because the symmetry is the same on the top and the bottom. If we divide by 4! on the top and bottom, it's like we don't care about it.

A3. You have 30 cards: 10 red, 10 green, and 10 blue. The cards of each are labelled 1, 2, 3, ..., 10. You randomly draw five cards from the 30. What is the probability that two different values will be repeated exactly two times each? An example of such a collection of five cards is red 4, green 4, blue 2, red 9.

First, remember:

$$P = \frac{\#wins}{\#total}$$

Let's do the $\#total$ first. This is just: $\binom{30}{5}$.

Now for the $\#wins$. We first need to choose the values that are repeated, $\binom{10}{2}$. Then the colors for each of those values, $\binom{3}{2}$ for each pair. Lastly, we need to choose the other value that is different from the other two $\binom{8}{1}$ and the suit for that card $\binom{3}{1}$. Multiplying these together gives us:

$$\binom{10}{2}\binom{3}{2}\binom{3}{2}\binom{8}{1}\binom{3}{1}$$

Our final answer is then:

$$P = \frac{\binom{10}{2}\binom{3}{2}\binom{3}{2}\binom{8}{1}\binom{3}{1}}{\binom{30}{5}}$$

A4. You have 30 cards: 10 red, 10 green and 10 blue. The cards of each color are labeled 1, 2, 3, ..., 10. You randomly draw 3 cards. The dealer tells you that at least one of the card values is not repeated. What is the probability that all three cards values are different?

First, we notice that we don't care about color. Next, we think about givens:

$$P(3 \text{ dif}| \geq 1\text{not repeated}) = \frac{P(3 \text{ dif} \cap \geq 1\text{not repeated})}{P(\geq 1\text{not repeated})} = \frac{P(3 \text{ dif})}{P(\geq 1\text{not repeated})}$$

We first recognize that:

$$P(\geq 1\text{not repeated}) = P(3 \text{ dif}) + P(2 \text{ match, } 1 \text{ dif})$$

The probability that all 3 are different is:

$$P(3 \text{ dif}) = \frac{\binom{10}{3}}{\binom{30}{3}}$$

The probability that 2 match, but not the third is:

$$P(2 \text{ match, } 1 \text{ dif}) = \frac{\binom{10}{1}\binom{9}{1}}{\binom{30}{3}}$$

The numerator doesn't have a factor of 2 anywhere because the first set (the pair) is distinguishable from the single, so we've already split those. We don't need to multiply by 2 to flip those two identities (we've already considered that) or divide by 2 becuase we can't tell the difference (because we can tell the difference).

Next, we see that the denominators are the same, so they're going to fall out. That gives us a final answer of:

$$P(3 \text{ dif}| \geq 1\text{not repeated}) = \frac{\binom{10}{3}}{\binom{10}{3} + \binom{10}{1}\binom{9}{1}}$$

A5. Suppose that you draw three cards from a fair 52 card deck. What is the probability that you draw a Jack of diamonds third given that the first two cards are diamonds?

This question is an A question, so it's subtly hard. Let's go back to our usual rules for givens:

$$P(3rd = JD|2D) = \frac{P(3rd = JD \cap 2D)}{P(2D)}$$

First, we figure out the probability of 2 diamonds. That's simple:

$$P(2D) = \frac{\binom{13}{2}\binom{50}{1}}{\binom{52}{3}}$$

Note that we're still choosing 3 cards, but we require the first 2 to be diamonds.

For the numerator, we need to think in two cases:

(1) We already have the Jack of diamonds in the first two cards.

(2) We can get the Jack of diamonds on the last card.

Note that in the first case (you already got the Jack), the probability of getting the Jack of diamonds on the third try is zero, so that case doesn't appear in the solution, but it does help us understand the second case. In the second case, we choose 2 diamonds that are not the Jack, giving us: $\binom{13-1}{2} = \binom{12}{2}$ choices of cards. We then must choose the Jack last, giving us no new combinations of possibilities. Therefore:

$$P(3rd = JD \cap 2D) = \frac{\binom{12}{2}}{\binom{52}{3}}$$

We again recognize that the denominator matches in both cases, so it'll divide out. That gives us a final answer of:

$$P(3rd = JD|2D) = \frac{\binom{12}{2}}{\binom{13}{2} \cdot 50}$$

A6. A disease has a prevalence of $\frac{1}{4}$ in a population. Test A gives 10% false positives and 2% false negatives. Test B gives 5% false positives and 20% false negatives. The false positive and negative rates for the tests depend only on whether or not the individual tested has the disease. What is the probability that you test positive on the second test (B), given that you test positive on the first test (A)?

This reminds me of homework again. In this case, it's a law of total probability problem hidden within what seems like a Bayes Theorem problem, but isn't. First, let's write out the given:

$$P(+_2|+_1) = \frac{P(+_2 \cap +_1)}{P(+_1)}$$

First, let's consider the denominator because it's easier. We know that:

$$P(+_1) = P(+_1|D)P(D) + P(+_1|D^c)P(D^c)$$

We recognize the first term as the true positive rate, which is 1-the false negative rate:

$$\begin{aligned} P(-_1|D) =& \text{false negative} = 0.02 \\ P(+_1|D) =& 1 - 0.02 = 0.98 \end{aligned}$$

The second conditional term is just the false positive rate, which was given to be 10%=0.1. Therefore, we get:

$$P(+_1) = 0.98\frac{1}{4} + 0.1\frac{3}{4}$$

Now for the numerator. We again use the law of total probability to say:

$$\begin{aligned} P(+_2 +_1) =& P(+_2 \cap +_1 | D) P(D) + P(+_2 \cap +_1 | D^c) P(D^c) \\ =& P(+_2|D) P(+_1|D) P(D) + P(+_2|D^c) P(+_1|D^c) P(D^c) \end{aligned}$$

Wait, we recognize that again. We need the true positive rate of the second test (test B), which is:

$$\begin{aligned} P(-_2|D) =& \text{false negative} = 0.2 \\ P(+_2|D) =& 1 - 0.2 = 0.8 \end{aligned}$$

And we read off the false positive rate as 5%. We plug that in to get:

$$P(+_2 +_1) = 0.8 \cdot 0.98 \cdot \frac{1}{4} + 0.05 \cdot 0.1 \cdot \frac{3}{4}$$

Plugging those into our original formula gives us:

$$P(+_2|+_1) = \frac{0.8 \cdot 0.98 \cdot \frac{1}{4} + 0.05 \cdot 0.1 \cdot \frac{3}{4}}{0.98\frac{1}{4} + 0.1\frac{3}{4}}$$

You see that we never needed Bayes formula? Wohoo!

A7. On average, it takes four trials for a certain coin to land on heads when tossed. What is the probability that the coin lands on heads at least three times in five tosses given that it lands on heads at least once?

Firstly, let's decompact the first sentence. It says that the expected number of trials till you get a heads is 4. We can either intuit that the probability of a heads on any given trial is $\frac{1}{4}$, or we can remember that waiting till a certain event is a Geometric random variable, which has mean of $E(X) = \frac{1}{p} = 4 \Rightarrow p = \frac{1}{4}$.

Next, let's go back to that handy dandy formula that we've used repeatedly. Let's write it out for this case:

$$\begin{aligned} P\left(3H | \geq 1H, n = 5, p = \frac{1}{4}\right) =& \frac{P\left(3H \cap \geq 1H | n = 5, p = \frac{1}{4}\right)}{P\left(\geq 1H | n = 5, p = \frac{1}{4}\right)} \\ =& \frac{P\left(3H | n = 5, p = \frac{1}{4}\right)}{P\left(\geq 1H | n = 5, p = \frac{1}{4}\right)} \end{aligned}$$

We can either do this the hard way or the easy way. Let's do it the easy way, using the 1- trick to get the at least. We now see that the number of heads in 5 trials is Binomial. Yes, I know that we talked about a Geometric distribution earlier, but now we've switched to Binomial because we set the number of trials. For the 1- trick, we say that (dropping the annoying givens):

$$P(\geq 1H) = 1 - P(0H)$$

We then remember our Binomial distribution function, which tells us:

$$\begin{aligned} P(3H) =& \binom{5}{3} \left(\frac{1}{4}\right)^3 \left(\frac{3}{4}\right)^2 \\ P(\geq 1H) =& 1 - \binom{5}{0} \left(\frac{1}{4}\right)^0 \left(\frac{3}{4}\right)^5 = 1 - \left(\frac{3}{4}\right)^5 \end{aligned}$$

We then plug that into our given formula to get:

$$P\left(3H| \geq 1H, n = 5, p = \frac{1}{4}\right) = \frac{\binom{5}{3}\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^2}{1 - \left(\frac{3}{4}\right)^5}$$

A8. Estimate the number of people you must randomly sample from a population to determine with an error of $\pm 1\%$ the prevalence of a disease so that the probability that you are correct within the specified error is at least 98%.

For one, this screams Chebychev and it also screams, "I did this in homework." First, let's revisit Chebychev's with the law of weak numbers:

$$P(|X - E(X)| \geq c) \leq \frac{Var(X)}{c^2} = \frac{1}{4nc^2} \text{ if we have Binomial trials}$$

Using what I showed you in a previous problem, we flip this to:

$$P(|X - E(X)| \leq c) \geq 1 - \frac{1}{4nc^2}$$

We are told that the probability that we're within the given error is 98%, which is the word version of saying:

$$P(|X - E(X)| \leq c) \geq 0.98$$

We are also told that we're within 1%, which says that

$$c = 0.01$$

When we plug in, we've got one equation at one unknown. We should know how to solve that:

$$\begin{aligned}
0.98 \geq& 1 - \frac{1}{4n \cdot 0.01^2} \\
0.02 \leq& \frac{1}{4n \cdot 0.01^2} \\
4 \cdot 0.01^2 \cdot 0.02 \leq& \frac{1}{n} \\
n \geq& \frac{1}{4 \cdot 0.02 \cdot 0.01^2} = 125,000
\end{aligned}$$

A9. Three people play Russian Roulette with a five shooter. Between each round, the position of the bullet is randomized. What is the probability that the player who goes third gets shot?

We can do this the hard way (see my previous homework with 2 players) or the easier way. Let's do this the easier way. We can say that the probability that the second player gets shot is:

$$P(2nd) = P(2nd|X)P(X) = P(1st)\frac{4}{5}$$

That is, if the first player doesn't get shot, X, then you start the game over with the second player acting like the first. We can do this again for the third player to say:

$$P(3rd) = P(1st)\frac{4}{5}\frac{4}{5}$$

We then remember that all probabilities add up to 1, so we say:

$$
\begin{aligned}
1 =& P(1st) + P(2nd) + P(3rd) \\
=& P(1st) + P(1st)\frac{4}{5} + P(1st)\frac{4}{5}\frac{4}{5} \\
1 =& P(1st)\left[1 + \frac{4}{5} + \frac{4^2}{5^2}\right] \\
=& P(1st)\left[\frac{5^2 + 5 \cdot 4 + 4^2}{5^2}\right] \\
\frac{5^2}{5^2 + 5 \cdot 4 + 4^2} =& P(1st)
\end{aligned}
$$

We then go back to our previous formula, that says:

$$
\begin{aligned}
P(3rd) =& P(1st)\frac{4}{5}\frac{4}{5} = \frac{5^2}{5^2 + 5 \cdot 4 + 4^2}\frac{4}{5}\frac{4}{5} \\
=& \frac{4^2}{5^2 + 5 \cdot 4 + 4^2}
\end{aligned}
$$

Of all the players, you want to be third.

A10. A bag contains 1 red ball, 1 green ball, and 1 yellow ball. You reach in the bag, take a ball, then put it back into the bag. You continue doing this until you get the same color ball twice in a row. What is the expected number of times you reach into the bag?

Now this is like the last question of the conditional expectation homework. If we're talking about one color ball, then we have, where X is not our color of interest:

$$
\begin{aligned}
E(T) =& E(T|RR)P(RR) + E(T|RX)P(RX) + E(T|X)P(X) \\
=& 2\frac{1}{3^2} + [2 + E(T)]\frac{1}{3}\frac{2}{3} + [1 + E(T)]\frac{2}{3}
\end{aligned}
$$

BUT, we don't care about the initial color of the ball. Whatever we get first, we try to match it, so the first pick doesn't matter. That changes it to (S indicates the color of the first ball, whatever it is):

$$
\begin{aligned}
E(T) =& E(T|SS)P(SS) + E(T|SD)P(SD) \\
=& 2\frac{1}{3^2} + [1 + E(T)]\frac{1}{3}\frac{2}{3}
\end{aligned}
$$

The astute of you would notice that $E(T|SD) = 1 + E(T)$ instead of $E(T|RX) = 2 + E(T)$. In this case, we only lost one toss becuase we instantly try to start matching our second toss with the third toss. This case of overlap gets really really confusing really fast. Solving that equation though:

$$
\begin{aligned}
E(T) =& 2\frac{1}{3^2} + [1 + E(T)]\frac{1}{3}\frac{2}{3} \\
\left[1 - \frac{2}{3^2}\right] E(T) =& \frac{2 + 1 \cdot 2}{3^2} \\
\frac{9 - 2}{3^2} E(T) =& \frac{4}{3^2} \\
E(T) =& \frac{4}{7}
\end{aligned}
$$

A11. The random variable X is uniformly distributed on [0,2] and the random variable Y is exponentially distbuted with parameter $\lambda = 5$. What is the probability that the minimum of X and Y is less than 1?

First, let's assume independence. Then, we recognize that one of the four possibilities can occur: both X and Y can be less than 1, one or the other can be less than 1 (two options), or both are greater than 1. We then can use:

$$\begin{aligned} P(Z < 1) =& P(Y < 1 \cap X < 1) = P(Y < 1)P(X < 1) \\ =& \left[1 - e^{-5z}\right]\left[1 - \frac{z}{2}\right] \\ =& \left[1 - e^{-5}\right]\left[1 - \frac{1}{2}\right] \\ =& \frac{1 - e^{-5}}{2} \end{aligned}$$

A12. A radioactive substance has a half life of 2 hours. Three atoms of this substance are in a box. A bomb goes off when the first atom decays. What is the expected length of time it takes for the bomb to go off?

At this point, you really wish that you had looked at the solutions to the homework that I gave out. If you're waiting for the first thing of 3 exponential processes, then it's like you're waiting for something that comes 3 times as fast. That means that the half life is now $\frac{2}{3}$ hours and you only have one atom. We have a handy formula (see homework solutions for derivation) for λ in that case:

$$\lambda = \frac{\ln 2}{t_{1/2}} = \frac{3 \ln 2}{2}$$

Now we also had memorized that:

$$E(T) = \frac{1}{\lambda} = \frac{2}{3 \ln 2}$$