Math3C: Homework 1

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You should NOT look at these answers before you attempt the problems. If you do, then you're not going to learn as well. Note that these answers are not reviewed by Dr. Weisbert and/or may include some errors. If you find one or more, let me know and I'll edit the document. My email is wesleytk@ucla.edu.

Exercise 1. Let $A = \{\{x\}, \{\{x\}\}, x, y, z, w, a, b\}$ and let $B = \{\{y, z\}, a, b, c, d\}$. Describe the sets $A \cup B$, $A \cap B$, $A \backslash B$, and $B \backslash A$.

- $A \cup B = \{\{x\}, \{\{x\}\}, x, y, z, w, a, b, \{y, z\}, a, b, c, d\} = \{\{x\}, \{\{x\}\}, x, y, z, w, a, b, \{y, z\}, c, d\}$
- $A \cap B = \{a, b\}$
- $A \backslash B = A - A \cap B = \{\{x\}, \{\{x\}\}, x, y, z, w, a, b, \{y, z\}, c, d\}$
- $B \backslash A = B - A \cap B = \{\{y, z\}, c, d\}$

Exercise 2. Let $A = \{\{x, a, b\}, \{\{a\}\}, x, z, a, b\}$ and let $B = \{\{a\}, \{a, b\}, c, d\}$. Describe the sets $A \cup B$, $A \cap B$, $A \backslash B$, and $B \backslash A$.

- $A \cup B = \{\{x, a, b\}, \{\{a\}\}, x, z, a, b, \{a\}, \{a, b\}, c, d\}$
- $A \cap B = \emptyset$
- $A \backslash B = A$
- $B \backslash A = B$

Exercise 3. Let $A = \{\{x, a\}, \{x\}, x, y\}$. List all the subsets of A.

(1) A

(2) $\emptyset$

(3) $\{\{x, a\}, \{x\}, x\}$

(4) $\{\{x, a\}, \{x\}, y\}$

(5) $\{\{x, a\}, x, y\}$

(6) $\{\{x\}, x, y\}$

(7) $\{\{x, a\}, \{x\}\}$

(8) $\{\{x, a\}, y\}$

(9) $\{\{x\}, y\}$

(10) $\{\{x, a\}, x\}$

(11) $\{\{x\}, y\}$

(12) $\{x, y\}$

(13) $\{y\}$

(14) $\{x\}$

(15) $\{\{x, a\}\}$

(16) $\{\{x\}\}$

You will soon learn that the number of sets is $\binom{4}{4}+\binom{4}{3}+\binom{4}{2}+\binom{4}{1}+\binom{4}{0} = \frac{4!}{4!0!}+\frac{4!}{3!1!}+\frac{4!}{2!2!}\frac{4!}{1!3!}+\frac{4!}{0!4!} = 1+4+\frac{4\cdot 3}{2}+4+1 = 1+4+6+4+1 \overset{\checkmark}{=} 16$

Exercise 4. Find a one-to-one and onto function from $\{1,2,3,4,5\}$ to the set $A = \{x, \{a,b\}, y, z, w\}$. How many elements does A contain?

There are many such functions. One is a function, f, such that $1 \to x$, $2 \to \{a,b\}$, $3 \to y$, $4 \to z$ and $5 \to w$. A contains 5 elements, one of which is a set in itself.

Exercise 5. Find a one-to-one and onto function from the natural numbers ($\mathbb{N} = \{1,2,3,\ldots\}$) to the integers ($\mathbb{Z} = \{\ldots,-3,-2,-1,0,1,2,3,\ldots\}$). What does the existence of such a function tell us about the size of the two sets.

Let x be from $\mathbb{N}$, then $f(x) = \left\{ \begin{array}{cc} \frac{x-1}{2} & \text{If } x \text{ is odd} \\ -\frac{x}{2} & \text{If } x \text{ is even} \end{array} \right\}$. This means that $\mathbb{N}$ is just as big as $\mathbb{Z}$. Infinity is a strange place.

Exercise 6. Suppose that $A = \{x,y,z,w,a,b,c,d\}$ and $B = \{z,b,c,e,f\}$ are both subsets of the universal set $U = \{a,b,c,d,e,f,g,h,i,u,v,w,x,y,z\}$. Verify de Morgan's laws.

De Morgan's laws, p4, state (1), $(\cup_{i\in I}A_i)^c = \cap_{i\in I}A_i^c$ and (2) $(\cap_{i\in I}A_i)^c = \cup_{i\in I}A_i^c$. That is complicated math language. Let's depack. In this setting, $A_1 = A$ and $A_2 = B$.

First, what is $\cup_{i\in I}A_i = A\cup B = \{a,b,c,d,e,f,w,x,y,z\}$. The complement of that is $\{g,h,i,u,v\}$. Now consider $A^c = \{e,f,g,h,i,u,v\}$ and $B^c = \{a,g,h,i,u,v,w,x,y\}$. The intersection of $A^c \cap B^c = \{g,h,i,u,v\}$. This matches above.

Secondly, consider $A\cap B = \{b,c,z\}$. The complement of that is $\{a,d,e,f,g,h,i,u,v,w,x,y\}$. Now the right side of the equation. We have A^c and B^c above. Their union is $\{a,e,f,g,h,i,u,v,w,x,y\}$. This matches above, therefore de Morgan's laws are confirmed.

Exercise 7. Find a partition for the set $A = \{a,b,c,d,e,\{f,g,h\},i,j,k\}$ that is a disjoint partition and then find another partition that is not disjoint.

- disjoint: $A_1 = \{a,b,c,d,e\}$ and $A_2 = \{\{f,g,h\},i,j,k\}$.
- not disjoint: $A_1 = \{a,b,c,d,e,i\}$ and $A_2 = \{\{f,g,h\},i,j,k\}$.

Exercise 8. Suppose that you have a group of four people: Alice, Bob, Cindy, and Don. Find a set in correspondence with the ways that the four people can shake hands if you do not care to remember who initiated the handshake. Find a set in correspondence with the ways that the four people can shake hands if you do care to remember who initiated the handshake.

In the former case, this is a set of sets. Each of the inner sets are handshakes, whereas the greater set is the set of those handshakes.

$$\{\{Alice, Bob\}, \{Alice, Cindy\}, \{Alice, Don\},$$
$$\{Cindy, Bob\}, \{Don, Bob\}, \{Cindy, Don\}\}$$

When you care about order, you can use the curved notation that indicates order.

$$\begin{aligned}\{(Alice, Bob), (Alice, Cindy), (Alice, Don),\\ (Bob, Alice), (Bob, Cindy), (Bob, Don),\\ (Cindy, Alice), (Cindy, Bob), (Cindy, Don),\\ (Don, Alice), (Don, Bob), (Don, Cindy)\}\end{aligned}$$

Exercise 9. Suppose that you have two identical four sided dice. Find a set in correspondence with the set of possible rolls of the two dice. Note that you cannot tell the two dice apart.

This set is: $\{\{1,1\},\{1,2\},\{1,3\},\{1,4\},\{2,2\},\{2,3\},\{2,4\},\{3,3\},\{3,4\},\{4,4\}\}$. Where the number indicates the side of the dice showing.

Exercise 10. Suppose you have a single four sided die. You roll the die two times. Find the set in correspondence with the set of possible rolls.

In this case, order matters, so you have:

$$\begin{aligned}\{&(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4),\\ &(2,1),(3,1),(4,1),(3,2),(4,2),(4,3)\}\end{aligned}$$

Note that I'm not double counting the doubles.

Exercise 11. A bag of fruit contains four apples, two oranges, and three pears. Find a set in correspondence with the bag of fruit. How many fruit are contained in the bag?

The set is $\{apple_1, apple_2, apple_3, apple_4, orange_1, orange_2, pear_1, pear_2, pear_3\}$. Note that the order of the set doesn't matter, so there are still $4 + 2 + 3 = 9$ fruit in the bag.

Exercise 12. Eight friends go to a pen store and each will buy exactly one pen. There are three types of pens to choose from. Find a set that is in correspondence with the friends' possible choices and then compute the number of possible choices that the friends can make.

In this case, we can think of this as an ordered set where each character is a friend. Each friend is unique, just as each position in the word is unique. Therefore, one possible set is $(1,2,3,1,2,3,1,2)$. Note the length is 8 and the range is one to 3.

To get the number of possible choices, notice that each friend has 3 choices. Then, we have 3 choices for each position, and we use the multiplication rule to get the number of choices as $3 \cdot 3 \cdot 3 \cdots 3 = 3^8 = 6561$.

Exercise 13. Eight friends go to a pen store and will buy exactly one pen. There are three types of pens to choose from: a red, green, and blue pen. Alice and Bob are two of the friends and they always compete for originality. They will never choose the same color of pen. How many choices are there?

Let's say we don't like Bob, so we're going to put Bob last. There are no conditions on the first seven friends, so they have 3^7 possibilities for their choices, as in (12). So, how many choices of pen does Bob have? Alice has already chosen, so he's got 2 choices. By the rule of multiplication, the total number of choices is $3^7 \cdot 2 = 4374.$. We never talk about order in this question, so order doesn't matter. If you've got questions about why we didn't do certain things, ask.

Exercise 14. Your job is to paint bookcases one of three different colors or to leave the bookcase unpainted. You have eight different bookcases and all book cases have a different height. In how many ways can you do your job?

Note that we can tell the difference between the book cases, so that's code for order matters. The next trick is that unpainted acts as a fourth color. Therefore, we just have 8 items, each item has 4 choices, and order matters so we use the rule of multiplication to get $4^8 = 65,536$.

Exercise 15. Your job is to paint bookcases one of three different colors or to leave the bookcases unpainted. You have eight different bookcases and all of the book cases have a different hieght. If the tallest bookcases is red, then the shortest bookcases must be painted the same color. In how many ways can you do your job?

This needs to be split into subsets. If the tallest bookcase is red, then all the bookcases being the same color acts as if they are one item. So if red, then 4 possibilities for the other ones.

Now, if the tallest bookcase isn't red, we have the multiplication rule again. We've only got 3 choices for the tallest bookcase but we're unconstrained for the rest, giving us: $3 \cdot 4^7 = 49,152$.

Remember we had 4 other possibilies? Let's add them together to get 49,156.

Exercise 16. On each school day, Monday through Friday, you have one hour free. During this free hour, you can study, lift weights or sleep. Since you want a varied schedule, you will not perform the same activity on consecutive days. If you lift weights on Wednesday, you must sleep on Thursday. There are no further restrictions. In how many ways can you plan your schedule?

This has a lot of subsets. Let's start with the most constrained case. If you lifted on Wednesday, then you sleep on Thursday. For Monday, you have 3 options, Tuesday you have 2 (because you can't lift) and Friday you have 2 (because you can't sleep). That gives you $3 \cdot 2 \cdot 2 = 12$ possibilities.

Now the less constraned case. Let's start at the complicated point: Wednesday. Let's suppose you studied on Wednesday. Then you have 2 choices for what to do on Tuesday and two on Thursday. Next, since you did something on Tuesday and Thursday, you've got another two choices on Monday and Friday. That means, under the rule of multiplication, you have 2^4 choices if you studied on Wednesday. The same goes for if you lifted on Wednesday, giving us another 2^4.

Now we have considered all possibilities of things that you did on Wednesday, so let's add each of those choices. $12 + 2^4 + 2^4 = 12 + 16 + 16 = 44$.

Exercise 17. In how many ways can eight people line up?

In this case, order matters so we want to use the multiplication rule, but we can't apply it as we usually did. Let's build the line from the beginning. At first, we have 8 possibilities for the first person in line. For the second position, we have 7 possibilities (because someone is already first). For the third, we have 6 possibilities. You see the pattern?

Overall, we have $8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 8! = 40,320$ different ways.

Exercise 18. In how many ways can eight people line up if three of the people, Alice, Bob, and Cindy can't stand next to each other?

We could do this, as we did above, as a series of cases. That would get really complicated really fast. Let's think of an easier way to do it.

Let's put Alice/Bob and Cindy in 3 spots to start with. For all the remaining people, we can put them into position however we please. We have 5 people and 5 positions, so for every combination of Alice/Bob/Cindy, we have 5!=120 positions of the other people.

Now let's figure out how many Alice/Bob/Cindy combinations we have, and then we're going to use the rule of multiplication to combine the two.

Imagine Alice is first, then we have 6 choices for where to put Bob. If Bob is on the end, we then have 4 choices for where to put Cindy. That's a total of 4 combinations so far. If Bob's not on the end, we have 3 choices. There are 4 non-ends to put Bob, so $4 \cdot 3$ gives us 12 more possibilities for a total of 16. Note that the labels of Alice/Bob and Cindy don't really matter. Let's think of the places Alice/Bob and Cindy are in as special positions that any of them can go into. We have 3!=6 ways to put them into those positions (3 people, 3 positions). Multiplying again, we get $16 \cdot 3 = 48$.

Now imagine that Alice wasn't first. Have we done this already? Yes, partially. Remember we switched the labels above, so if Bob wasn't on the end, then Alice wasn't on the end. However, let's rephrase. What if no one was on the ends? By that, I mean that none of our named people were on the end. We haven't considered that yet. There are 6 internal positions (non-ends). Put Alice at one of the sides of the internal positions, let's call it position 2. That leaves 4 allowed positions for Bob that are not ends. If he choses one as close as possible to Alice, then Cindy has 3 options for where to go. That gives us 3 more possibilities. If he choses one farther away (there are 3 of those), Cindy has 2 options. So multiply $3 + 3 \cdot 2 = 9$. Then, multiply that by 3! because Alice/Bob and Cindy can be shuffled. That gives us $9 \cdot 3! = 54$.

Are we done with all the cases? Yes. There's no way for none of our special people to be in neither position 1, 2, 7 or 8 without violating our conditions. Okay, let's finally add the two cases of special people: 48+54=102. Then multiply by 120 because of our first paragraph to get 12,240. Note that this is smaller than 40,320, so it checks out as a sane answer.

Final answer: 12,240 ways.

Exercise 19. How many ways can a frog jump on seven lilly pads if it jumps on one lilly pad exactly once? The lilly pads all look different and the frog starts off in the water.

This is the same as question 17, just with frogs instead of people. Thereby, the answer is $7! = 5,040$.

Exercise 20. How many ways can a frog jump on seven lilly pads if it jumps on one lilly pad exactly two times and all other lilly pads exactly once? The lilly pads all look different and the frog starts off in the water.

Let's start with the previous question, then figure out where to put that extra jump. We have at least $5,040$ jumps. For each sequence of jumps, think about where you can put the other jump. If we start with jump sequence 1234567, we can put the next jump at positions, x, giving us: $x1x2x3x4x5x6x7x$. There are 8 such positions. So, you want to say that our answer is $8 \cdot 5,040$, but if you did, then you missed something.

If we jump on lilly pad 1 twice, then this sequence $x1234567$ and $1x234567$ look the same. We always jump on lilly pad 1 during each sequence, so if we start with a jump sequence, say 1234567, there are 7 spots we can jump to make a unique new sequence (the first two spots look the same). That gives us $7 \cdot 5,040 = 35,280$.

Exercise 21. How many ways can a frog jump on seven lilly pads if it jumps on one lilly pad exactly two times and all the other lilly pads exactly once? This time, however, it cannot make consecutive jumps on the same lilly pad- a more realistic scenario! Once again, the lilly pads all look different and the frog starts off in the water.

We're already done the work for this in the previous question. Yay! Consider our jump sequence of 1234567. There are 6 spots where you can repeat lilly pad 1 so that you don't have a repeat: $12x3x4x5x6x7x$. So we're going to multiply $6 \cdot 5,040$. But way, let's make sure there's not an edge effect. If our sequence is 2134567, then we have six spots still: $x213x4x5x6x7x$. So there are no edge effects and we get $6 \cdot 5,040 = 30,240$.

Exercise 22. How many ways can eight people stand in a circle?

This gets really complicated really fast. Let's assume for simplicity that we know the difference between left and right (i.e. $12345678 \neq 87654321$). We know that it's less than 8! because a circle of 12345678 is the same as 23456781. That's called a cyclic permutation. For each sequence of people, how many cyclic permutations do we have to make an equivalent sequence: 8. That means that for each linear sequence, the circle makes 8 of them redundant. That gives us $8!/8 = 7! = 5,040$ possible ways for eight people to stand in a circle.

Exercise 23. Suppose that you have five colors to choose from. In how many ways can you paint the vertices of a square if vertices connected by an edge must be painted different colors? In how many ways can you pain the vertices of a pentagon if vertices connected by an edge must be painted different colors?

Let's do the square first. Let's label the edges in clockwise order, 1234. We have 5 choices for the edge 1, and then 4 for the edge 2 because edge 1 is painted. We still have 4 choices for edge 3, because of edge 2. Lastly, we have 3 choices for edge 4 because it's connected to both edge 1 and 3, both of whom are painted. Using the rule of multiplication, we then have $5 \cdot 4 \cdot 4 \cdot 3$ choices. If you say yes, you forgot Exercise 22. We can rotate the square 4 times and it'll all be the same. That gives us $\frac{5 \cdot 4 \cdot 4 \cdot 3}{4} = 60$ choices.

Now the pentagon. It's the same thing: 5 choices for edge 1, 4 for edge 2, 4 for edge 3, 4 for edge 4 and 3 for edge 5. There are 5 ways to rotate a pentagon, so we have to divide by 5. This gives us $\frac{5 \cdot 4 \cdot 4 \cdot 4 \cdot 3}{5} = 192$ choices.

Exercise 24. Suppose that you have 10 points marked on a piece of paper. How many arrows can you draw between the points if each arrow has one point at its head and a different point at its tail?

Let's start at point 1. It has 9 choices of where the arrow can go if the arrow head is somewhere else and the tail is at point 1. Next, consider point 2. It's the same thing. The question specifies that tails and heads of arrows are different, so arrow $1 \to 2$ is different from $2 \to 1$. Using the rule of multiplication, we get $9^{10} = 3,486,784,401$.