Math3C: Homework 2

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You should NOT look at these answers before you attempt the problems. If you do, then you're not going to learn as well. Note that these answers are not reviewed by Dr. Weisbert and/or may include some errors (as we figured out first week). If you find one or more, let me know and I'll edit the document. My email is wesleytk@ucla.edu.

Exercise 1. There are 40 people at a party. Each person shakes every other person's hand (right hand to right hand) exactly one time. How many handshakes occur?

For each handshake, you choose 2 people from the 40. So, the question is how many choices of 2 people can you make from 40 people

$$\binom{40}{2} = \frac{40!}{2!38!} = 20 \cdot 39 = 780$$

Exercise 2. You have a bag with 10 marbles all of different colors. You reach your hand in the bag and take four marbles. How many possible collections of four marbles can you gather?

This has no tricks to it, it's just 10 choose 4:

$$\binom{10}{4} = \frac{10!}{4!6!} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2} = 10 \cdot 3 \cdot 7 = 210$$

Exercise 3. A bookshelf has 10 math books and 12 physics books. You are to select five math books and four physics books to read. In how many ways can you make your selection?

This is the combination of three problems. (1) How many different math selections can you make:

$$\binom{10}{5} = \frac{10!}{5!5!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{5 \cdot 4 \cdot 3 \cdot 2} = \frac{10 \cdot 9 \cdot 2 \cdot 7}{=} 252$$

(2) How many different physics selections can you make:

$$\binom{12}{4} = \frac{12}{4!8!} = \frac{12 \cdot 11 \cdot 10 \cdot 9}{4 \cdot 3 \cdot 2} = 495$$

(3) How do you combine those two selections? The multiplication rule: $252 \cdot 495 = 124,740$.

Exercise 4. You line up 20 people for a photograph. Three of the people are identical triplets wearing the exact same clothing. All other individuals are distinguishable in the photograph. How many photographs are possible?

Let's assume that the triplets don't have moles or anything to tell them apart either. To line up 20 people, you have 20! possibilities. For each of those possibilities, you can flip the three triplets around 3! ways that you can't tell the difference between. That means that you're overcounting each possibility 3! times. That leads us to:

$$P(20, 17) = \frac{20!}{3!}$$

Exercise 5. You have 10 rings. Three rings are solid gold bands and look identical. All other rings look different. You select five rings to place on your right hand then take a photograph of your hand. How many distinct photgraphs can you take?

Let's assume that you take the photograph from the same angle and lighting each time. Let's also assume that the order with which you place the rings on each individual finger doesn't matter (because that would get crazy very quickly).

First, we consider the different sets of 5 rings that we could select, before we put them on. Here are the number of different ways we can select the other rings, when we have:

- 0 gold: $\binom{7}{5}$ because we have to choose 5 rings from 7 different rings. We don't care about the order of our selection.
- 1 gold: $\binom{7}{4}$ because we have to choose 4 rings from 7 different rings. We've already chosen 1 ring (the gold ring). Note that we don't care about if we're selecting different gold rings.
- 2 gold: $\binom{7}{3}$
- 3 gold: $\binom{7}{2}$

Next, consider the different ways to put the rings on that we have selected. First, suppose that we put on the p different rings first. Then we have 5 options for each ring, and we can distinguish each ring, giving us 5^p options.

Now, consider how many ways we can put on the gold rings after that. This is the Lesson 6 problem. n is the number of gold rings and k is the number of fingers, 5. We have $\binom{n+k-1}{k-1}$ choices.

Splitting this by cases, we get:

0 gold	1 gold	2 gold	3 gold
$\binom{7}{5}5^5$	$\binom{7}{4}5^4\binom{1+5-1}{5-1}$	$\binom{7}{3}5^3\binom{2+5-1}{5-1}$	$\binom{7}{2}5^2\binom{3+5-1}{5-1}$

Add the cases all together to get the answer.

Exercise 6. How many words can you make using all the letters of the word AAABBCCC-CCD?

This problem can be viewed in a lot of ways: either a permutation or multinomial. Let's do it the most general way. The word is 11 letters long. That gives us 11! possibilities. However, we have 3 A's, 2 B's and 5 C's. Each of those letters can be permuted 3!, 2! and 5! ways, respectively. Each of those permutations leads to overcounting. Overall, when we correct for that overcounting, we get:

$$\frac{11!}{5!3!2!}$$

Exercise 7. A flea stands at point 0. The flea can jump one integer to the left and one integer to the right each time he jumps. Therefore, the first time the flea jumps he may land on -1 or 1. The second time he jumps, he can leand on -2, 0, or 2, and so forth. Suppose the flea makes 20 jumps. How many ways can the flea get to 4 on his last jump? How many ways can the flea get to 5 on his last jump?

Let's do the easy problem first: how many ways can he get to 5? The answer is 0. Notice with odd numbers of jumps, you only get to odd numbers. With even numbers of jumps, you can get to even numbers only.

Now the harder problem. You've got 20 jumps and 2 possibilities for each jump. You want 4 more right jumps than left jumps. That's the same thing as choosing 12 right jumps and 8 left jumps. Wait, we know how to do that:

$$\binom{20}{12} = \frac{20!}{12!8!}$$

Exercise 8. A bag contains a red ball and a green ball. You reach your hand in the bag, take out a ball, observe its color, and put it back in the bag. You do this 12 times. In how many ways can you take a green ball out of the bag exactly 5 times?

You've only got 2 possibilities, which makes this problem easier. You are really asked, how many different possibilities do you have if you have 5 G's and 7 R's in different orders. We did that problem already (see Exercise 6). It's just:

$$\frac{12!}{7!5!}$$

Exercise 9. A bag contains a red ball and a green ball. You reach your hand in the bag, take out a ball, observe its color, and put it back in the bag. You do this 14 times. In how many ways can you take a green ball out of the bag at least 3 times?

There is a long way to do this problem and a short way. The long way, use the trick we said in Exercise 8 and do:

$$\frac{14!}{3!11!} + \frac{14!}{4!10!} + \cdots + \frac{14!}{13!1!} + \frac{14!}{14!0!}$$

That is the number of possibilities when we take the green ball out exactly 3, 4, ..., 13 or 14 times.

The short way is only slightly shorter. Recognize that $2^{14} = \sum_{k=0}^{14} \binom{14}{k}$. (see `http://en.wikipedia.org/wiki/Combination#Number_of_k-combinations_for_all_k`) Then, we can express at least 3 times as:

$$2^{14} - \frac{14!}{0!14!} - \frac{14!}{1!13!} - \frac{14!}{2!}12!$$

That latter sum is much easier to count. The answer will be identical.

Exercise 10. A bag contains a red ball, a green ball, and a yellow ball. You reach your hand in to the bag, take out a ball, observe its color, and put it back in the bag. You do this 12 times. In how many ways can you take out 3 red balls, 4 green balls, and 5 yellow balls in 12 trials?

First check question zero: does $3+4+5=12$? Yes. Okay. Now refer to the trick we did in Exercise 6 and 8. It's just a sequence of 3 R's, 4 G's and 5 Y's. That's:

$$\frac{12!}{3!4!5!}$$

Exercise 11. What is the coefficient of x^3y^2 in the expansion $(x+y+z+3)^8$?

If this was just $(x+y)^8$, then we'd just use $B(8,3) = \binom{8}{3}$, but we've got that $z+3$ involved. First to deal with the z. Note that z doesn't appear, so let's assume (for now) it's order is 0, giving us the coefficient of $x^3y^2z^0$, which is again $B(8,3)$. Now what about the 3? Note that $3+2+0=5<8$. However, each term in $(x+y+z+3)^8$ needs to be of order 8. That means that we have 3 more powers of 3 involved. The order of those 3's, however, also doesn't matter, so we need to divide by 3!.

Overall, we multiply the two to get: $3^3\frac{8!}{3!2!3!}$. But what if the power of z was not zero? That also counts as a coefficient.

The next terms are built in the same way:

$$x^3y^2\left[3^3\frac{8!}{3!2!3!}+z3^2\frac{8!}{3!2!2!}+3z^2\frac{8!}{3!2!2!}+z^3\frac{8!}{3!2!3!}\right]$$

Exercise 12. A class has 20 students. There is exactly one pair of identically dressed twins in the class. There are also three identically dressed triplets in the class. You randomly select 7 people for a photograph. How many distinct photographs are possible?

This can either be split into zillions of different cases, or we can think of how we derived the combinatorial/permutation formula. Let's do the cases first.

Case 1: Two full sets of triplets are in the photo. You then have $\frac{7!}{3!3!}$ options. You also have $\binom{3}{2}$ ways of selecting which sets of triplets are allowed. You also have 13 choices for that random person. In total:

$$\frac{7!}{3!3!}\binom{3}{2}\cdot 13$$

Case 2: One full set of triplets, one with 2/3 of a triplet, and the twins. You have $\frac{7!}{3!2!2!}$. You also have $\binom{3}{2}$ ways of selecting which sets of triplets are allowed (note that we don't care about which of the 2/3 triplets we take). In total:

$$\frac{7!}{3!3!}\binom{3}{2}$$

Case 3: One full set of triplets, the twins, one with 1/3 of a triplets, and a random person. You have $\frac{7!}{3!2!}$. You also have $\binom{3}{2}$ ways of selecting the triplets and 13 ways of selecting the random. In total:

$$\frac{7!}{3!2!}\binom{3}{2}\cdot 13$$

Case 4: One full set of triplets, the twins, and 2 random people. You have $\frac{7!}{3!2!}$. You have 3 choices for the triplet set and 13 choose 2 choices for the random. In total:

$$\frac{7!}{3!2!}\cdot 3\cdot\frac{13!}{11!2!}$$

Case 5: One full set of triplets, plus 4 random people. You have $\frac{7!}{3!}$. Again, 3 choices for the triplet set and $C(13,4)$ choices for the random. In total:

$$\frac{7!}{3!}\cdot 3\cdot\frac{13!}{4!9!}$$

Case 6: The twins, plus 5 random people. You have $\frac{7!}{2!}$ orders and $C(13,5)$ choices for the random. In total:

$$\frac{7!}{2!}\frac{13!}{5!8!}$$

Case 7: 7 random people. You have 7! orders and $C(13,7)$ choices for the random. In total:

$$7!\cdot\frac{13!}{7!6!}$$

Note that when we were selecting the randoms, we didn't care about the order because we already took that into account when ordering them with the twins/triplets.

We then add all the cases together and get:

$$\frac{7!}{3!3!}\binom{3}{2}\cdot 13+\frac{7!}{3!3!}\binom{3}{2}+\frac{7!}{3!2!}\binom{3}{2}\cdot 13+\frac{7!}{3!2!}\cdot 3\cdot\frac{13!}{11!2!}+\frac{7!}{3!}\cdot 3\cdot\frac{13!}{4!9!}+\frac{7!}{2!}\frac{13!}{5!8!}+7!\cdot\frac{13!}{7!6!}$$

This is the final answer.

Let's simplify this:

$$7!\left[\frac{3!}{3!3!2!}\cdot 13+\frac{3!}{2!3!3!}+\frac{3!}{3!2!2!}\cdot 13+\frac{1}{3!2!}\cdot 3\cdot\frac{13!}{11!2!}+\frac{1}{3!}\cdot 3\cdot\frac{13!}{4!9!}+\frac{1}{2!}\frac{13!}{5!8!}+\frac{13!}{7!6!}\right]$$

$$7!\left[\frac{13}{3!2!}+\frac{1}{2!3!}+\frac{13}{2!2!}+\frac{13\cdot 12}{2!2!2!}+\frac{13!}{4!9!2!}+\frac{13!}{2!5!8!}+\frac{13!}{7!6!}\right]$$

$$\frac{7!}{2\cdot 2}\left[\frac{13}{3}+\frac{1}{3}+13+13\cdot 6+\frac{13!}{4\cdot 3\cdot 9!}+\frac{13!2!}{5!8!}+\frac{13!}{7!6!}\right]$$

You see that it doesn't simplify? That means that any shortcut doesn't work.

Here's the logic behind the INCORRECT shortcut. Let's first ignore that we have 7 people chosen for the photograph. How many photographs do we have of the 20 students? It's

$$\frac{20!}{2!3!3!3!}$$

Now we only really care about the first 7 people in line, so there are 13 people that we don't care about the order. That means that we get:

$$\frac{20!}{2!3!3!3!13!}$$

But then we worry if we double divided. What if when we divided for the triples/twins, we already took out some of the pictures that we wanted to take out when we just cared about the first 7.

The shortcut way doesn't match the zillions of cases way, so the shortcut is wrong, for the reason I explained above.

Exercise 13. A class has 20 students. You will take 6 people from the group of 20 and line them up for a photograph. Alice and Bob are in the class and they detect each other. They will not stand next to each other in the photograph. How many photographs can be taken.

This is ideally split into cases. First, we don't choose either Alice or Bob. That means that we need to choose 6 people from 18, then photograph them. That's $\frac{18!}{12!}$.

Next, let's choose Alice and put her in one of 6 positions. Then we need to choose 5 people from 18 for the other positions. That's $6\cdot\frac{18!}{13!}$.

Next, let's choose Bob. That's another $6\cdot\frac{19!}{14!}$.

Lastly, let's choose Alice and Bob. Together. This is a problem that we did last week. There are 9 allowed positions of Alice and Bob (Alice first, then Bob has 4 options; Alice second, then Bob has 3 options, etc.). We can then flip Alice and Bob, giving us 2! more ways. Now we need the other 4 people. That's a permutation $P(18,4)=\frac{18!}{14!}$. Multiplying that all together we get: $9\cdot 2\cdot\frac{18!}{14!}$.

Adding all those possibilities gives us:

$$\frac{18!}{12!} + 2 \cdot 6 \cdot \frac{18!}{13!} + 9 \cdot 2 \cdot \frac{18!}{14!}$$

Exercise 14. Suppose that a class has 10 students. Alice is one of the 10 students. On any given day, students may or may not come to class. How many different groups of students can come to class if we know that Alice always shows up?

If we know Alice is always there, then she doesn't count. This is just the sum of all the possible groups that can show up: $\sum_{k=0}^{9} \binom{9}{k}$. Wait, we know what that is: 2^9 (see Exercise 9).

Exercise 15. You have seven types of soda and are meant to bring a collection of 12 sodas to a party. How many different collections can you bring?

If you're confused, see Lesson 6. The types of soda are the rings to put your fingers on, k, whereas the 12 sodas are the rings that you're putting in each bin, n. Note that you can tell the difference between the different sodas, but you can't tell the difference between the 'counts' of sodas. If you try to do it the other way, you run into trouble because you don't know how many total sodas you can choose from in the seven types. When you run into infinity, run away.

Then we just apply the formula:

$$\binom{n+k-1}{n} = \binom{12+7-1}{12} = \binom{18}{12} = \frac{18!}{12!6!}$$

Exercise 16. You have seven types of soda and are meant to bring a collection of 12 sodas to a party. How many different collections can you bring if you must bring exactly 3 cokes?

This is the same question as Exercise 15, except you've already had a few chosen for you. The total number of sodas you have a choice on is 9. BUT, you can't bring any more cokes, so you only have 6 options for types of soda. Otherwise, you're doing the same problem, so let's do the same thing we did before:

$$\binom{n+k-1}{n} = \binom{9+6-1}{9} = \binom{14}{9} = \frac{14!}{9!5!}$$

Exercise 17. You have seven types of soda and are meant to bring a collection of 12 sodas to a party. How many different collections can you bring if you must bring at least 3 cokes?

This is, again, the same question as Exercise 15, except you've limited the number of sodas you have a choice on to 9, but you still allow cokes. That means $n = 9$ and $k = 7$. That gives us:

$$\binom{n+k-1}{n} = \binom{9+7-1}{9} = \binom{15}{9} = \frac{15!}{9!6!}$$

Exercise 18. You have seven types of soda and are meant to bring a collection of 12 sodas to a party. How many different collections can you bring if you must bring at least 3 of one type?

This seems like the same thing as Exercise 17, except now you have a choice of which type you can bring at least 3 of. For each type we choose, we have $\frac{15!}{9!6!}$ different possibilities. We then have 7 different possibilities for the choice of the 'special' type of soda. That would suggest we have $7 \cdot \frac{15!}{9!6!}$ Different possibilities.

This is NOT the case. What if we said we need to bring at least 3 cokes and just so happened to bring at least 3 sprites too? We'd then double count those possibilities when we said that we need to bring 3 sprites. So how do we fix that? We split into cases.

Case 1: 3 each of 4 types of soda. That totals 12 soda cans. We have $C(7,4)$ choices of 4 soda types because we don't care about order.

Case 2: 3 each of 3 types of soda, then 2 matching types and 1 solo. We have $C(7,2)$ choices for the types with 3 each because they are indistinguishable past knowing that they have 3 each. For the remaining 2 types, we have $5 \cdot 4$ types.

Case 3: 3 each of 3 types of soda, then 3 solo's. We have $C(7,3)$ for the types with 3 each, and $4 \cdot 3 \cdot 2$ choices for the solo's.

Case 4: 3 each of 2 types of soda, 2 each of 3 types of soda. We have $C(7,2)$ choices for the types with 3 each and $C(5,3)$ choices for the types with 2 each.

Case 5: 3 each of 2 types of soda, 2 each of 2 types of soda, and 2 solo's. We have $C(7,2)$ for the types with 3 each, $C(5,2)$ for the types with 2 each, and $3 \cdot 2$ for the solo's.

Case 6: 3 each of 2 types of soda, 2 each of 1 type, and 4 solo's. We have $C(7,2)$ for the types with 3 each, 5 choices for the type with 2 options, and $4 \cdot 3 \cdot 2 \cdot 1$ for the solo's.

Case 7: 3 each of 2 types of soda, 6 solo's. Wait. We can't do this. That requires 8 types of soda.

Case 8: 3 each of 1 type of soda, 2 each of 4 types of soda, 1 solo. We have 7 choices for the types with 3 each, $C(6,4)$ choices for the types with 2 each, and 2 choices for the solo.

Case 9: 3 each of 1 type of soda, 2 each of 3 types of soda, 3 solo's. We have 7 choices for the types with 3 each, $C(6,3)$ choices for the types with 2, and $3 \cdot 2 \cdot 1$ choices for the solo's.

Case 10: 3 each for 1 type of soda, 2 each for 2 types of soda, 5 solo's. We also can't do this. It requires 8 types of soda.

Finally, let's add together all the cases:

$$\frac{7!}{4!3!} + \frac{7!}{2!5!} \cdot 5 \cdot 4 + \frac{7!}{3!4!} \cdot 4 \cdot 3 \cdot 2 + \frac{7!}{2!5!}\frac{5!}{3!2!} + \frac{7!}{2!5!}\frac{5!}{3!2!} \cdot 3 \cdot 2 + \frac{7!}{2!5!} \cdot 5! + 0 + 7 \cdot \frac{6!}{4!2!} \cdot 2 + 7 \cdot \frac{6!}{3!3!} \cdot 3! + 0$$

Let's simplify:

$$7!\left[\frac{1}{4!3!} + \frac{1}{2!3!} + \frac{1}{3!} + \frac{1}{2!3!2!} + \frac{1}{2!2!} + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{3!}\right]$$

$$7!\left[\frac{1}{12 \cdot 3!} + \frac{3}{12} + \frac{1}{12} + \frac{1}{4} + \frac{1}{2} + \frac{1}{12} + \frac{1}{6}\right]$$

$$7!\left[\frac{1}{12 \cdot 3!} + \frac{5}{12} + \frac{1}{4} + \frac{1}{2} + \frac{1}{6}\right]$$

$$7!\left[\frac{1}{12 \cdot 3!} + \frac{5}{12} + \frac{3}{12} + \frac{6}{12} + \frac{2}{12}\right]$$

$$7!\left[\frac{1}{12 \cdot 3!} + \frac{16}{12}\right]$$

$$7!\frac{16 \cdot 6 + 1}{12}$$

This brings us to our final answer:

$$\frac{97 \cdot 7!}{4!}$$

Exercise 19. How many ways can you write 500 as the sum of three distinct natural numbers?

Let's assume the order with which you write the sum doesn't matter. This seems like it's the same as the main question in Lesson 6, but it's not entirely. Let's start with that solution. You need 3= k natural numbers, so each number is a finger with which you'll put rings on. There's at least 1 ring on each finger because natural numbers are greater than 0. That gives us 497= n possibilities for where to put the other rings. That tells us we have:

$$\#(O) = \binom{497+3-1}{497} = \binom{499}{497}$$

BUT, we are including non-distinct possibilities and $\#(O)$ is inhomogenously ordered. Just as we did in Lesson 6, we need to split into three possibilities:

$$\Omega_1 = \frac{1}{3!}\#(A) + \frac{1}{3}\#(B) + 1\#(C)$$

Where set A is the ordered set of the unordered set we want, Ω_2, the number with 3 distinct natural numbers; B has one repeated number; C has all three numbers identical; and Ω_1 is the combinations of natural numbers (distinct or not). Note that for C to work, then $3 \cdot n_C = 500$ but 500 is not evenly divisible by 3, so $\#(C) = 0$. Note that ordered elements of A have 3! different orderings, which we view as all identical, which is why we divide by 3!. Similarly, the ordered elements of B have 3 orderings (places for the non-matching number), so we divide by 3.

To count $\#(A)$, we have the same thing as the book:

$$\#(A) = \#(O) - \#(B)$$

To count $\#(B)$, we do the same thing as the book. Suppose that elements in B can be written

$$b_1 + b_1 + b_2 = 2b_1 + b_2 = 497$$

Note we're dealing with the number of extra rings we put on the fingers, not the total number of rings on the fingers. We know that $b_2 \geq 0$, so we get

$$2b_1 \leq 497$$

This can be simplified to

$$\begin{aligned} 0 \leq & 2b_1 \leq 498 \\ 0 \leq & b_1 \leq 249 \end{aligned}$$

That tells us how many different possible values b_1 can take: 250. Let's think of examples to make sure we got the boundaries right. If $b_1 = 0$, then we would have $1+1+(b_2+1) = 500 \Rightarrow b_2 = 498-1 = 497$. Now set $b_1 = 249$, then we would have $(249+1)+(249+1)+(b_2+1) = 500 \Rightarrow b_2 = 0-1 = -1$. Wait. That doesn't work. That means that $b_1 \leq 248$.

So now we know that $\#(B) = 3 \cdot 249$.

Plugging this in, we have:

$$\begin{aligned}\Omega_2 =& \frac{1}{3!}\#(A)\\ \Omega_2 =& 3!\left(\#(O) - \#(B)\right)\\ =& \frac{1}{3!}\left(\binom{499}{497} - 3 \cdot 249\right)\end{aligned}$$

Note this is different from Lesson 6 because we do NOT want to count the elements in set B because they are not distinct.