Math3C: Homework 3
by Wesley Kerr (TA: Section 3a and 3b)

You should NOT look at these answers before you attempt the problems. If you do, then you're not going to learn as well. Note that these answers are not reviewed by Dr. Weisbert and/or may include some errors (as we figured out first week). If you find one or more, let me know and I'll edit the document. My email is wesleytk@ucla.edu.

Exercise 1. How many ways can you form a team of 3 people, a team of 4 people, and a team of 5 people from a collection of 25 people? People can serve on at most one team at a time since the teams will play against each other.

This is straight from the book:

$$\binom{25}{3}\binom{22}{4}\binom{18}{5}$$

Exercise 2. A certain Junior High School gym class has 20 students enrolled in it. Students will form two basketball teams of five people taken from the 20 students. The two teams will play against each other. One team is the green team and the other is the gold team. How many ways can the two teams be formed?

Let's start with choosing the two teams:

$$\binom{20}{5}\binom{15}{5}$$

Then let's think about if the two teams are distinguishable. Let's imagine that the first team we chose was the green team. If we flipped the students on the gold team and the green team, would we get a new way that the teams could be formed? YES, becuase one was green and one was gold. If they weren't labelled, then we'd need to divide by 2! because flipping the team labels would result in the same selection (aka overcounting).

Exercise 3. The chess club will have 5 members. The computer club will have 5 members. The math club will have 8 members. The physics club will have 9 members. There are 15 students in the class from which the members of the clubs will be chosen. How many possible ways can you assign the students to the clubs?

Now that you've done the first 2 problems, you've fallen into a pattern, which will get broken right now. The key difference: students can be in more than one club, so the number of students that you're choosing from doesn't decrease as you choose clubs. The answer is then:

$$\binom{15}{5}\binom{15}{5}\binom{15}{8}\binom{15}{9}$$

Order does matter because each club has a different label.

Exercise 4. You go to the park to play a pickup game of basketball. A total of 15 people show up and two teams of 6 will be selected. In how many ways can you form the two teams?

In this case, once you've chosen the teams, you could flip them and it wouldn't matter (becuase they're not labelled). See exercise 2 for how this changes things to:

$$\binom{15}{6}\binom{9}{6}\frac{1}{2!}$$

Exercise 5. You have two identical flash drives and you copy six files from a computer onto each of the flash drives. The computer contains 25 files. How many collections of flash drives can you produce?

This is straight from the book with different numbers. The answer is:

$$\binom{25}{6} + \binom{\binom{25}{6}}{2}$$

Exercise 6. You have 3 identical flash drives and you copy six files from a computer onto the flash drives. The computer contains 25 files. How many collections of flash drives can you produce?

This is also straight from the book, but I'm going to use the more general form. Think of the $\binom{25}{6}$ different flash drives as fingers, k. The number of flash drives is the number of identical rings. Therefore, you get:

$$\binom{\binom{25}{6}+2}{2}$$

Exercise 7. You have 10 identical flash drives and you copy six files from a computer onto each of the flash drives. The computer contains 30 files. How many collections of flash drives can you produce?

This is now the same as the last problem:

$$\binom{\binom{30}{6}+9}{9}$$

Exercise 8. A class of 20 students has 6 possible clubs that it can form and it is allowed to form two clubs of those 6. It is possible that both clubs are of the same type, that is, the clas may have two math clubs. Each club must contain exactly 5 members and members may simultaneously be in both clubs. How many ways can the class form the two clubs?

The key challenge in this question is that the clubs are not labelled. We need to worry about when they are distinguishable and when they aren't. So, this is going to split into 4 cases:

Case 1: The club type and club members are all the same (clubs are fully indistinguishable). We then just need to choose one club (6 possible) and one set of students (there are $\binom{20}{6}$ sets). Multiply those to get:

$$6 \cdot \binom{20}{6}$$

Case 2: The club type is the same, but the members in the club are different, making the clubs distinguishable. We then need to choose one club name (6 people), but we need to choose two different sets of students. There are $\binom{20}{6}$ sets of students, so we choose 2 of those to get a total of:

$$6 \cdot \binom{\binom{20}{6}}{2}$$

Case 3: The club types are different, but the members of the club are the same, making the clubs distinguishable. There are $\binom{6}{2}$ choices of club types, and $\binom{20}{6}$ choices of people in the club. That gives us:

$$\binom{6}{2}\binom{20}{6}$$

Case 4: Both the club names and members in the club are different. There are $\binom{6}{2}$ choices of 2 different clubs from 6 possible. Then, again, we must select 2 sets of students from $\binom{20}{6}$ sets of students, but because the club names are different, the order with which we choose those sets of students matters. That gives us a total of:

$$\binom{6}{2}\binom{20}{6}\cdot\left[\binom{20}{6}-1\right]$$

The final answer is then:

$$6\cdot\binom{20}{6}+6\cdot\binom{\binom{20}{6}}{2}+\binom{6}{2}\binom{20}{6}+\binom{6}{2}\binom{20}{6}\cdot\left[\binom{20}{6}-1\right]$$

Exercise 9. You roll a fair six sided die with sides labeled 1-6. What is the probability that the die lands on a number larger than 4?

This is just number of wins over number of possible combinations. There are two numbers above 4 (aka 5 and 6). There are 6 numbers total, giving us:

$$P(event)=\frac{2}{6}=\frac{1}{3}$$

Execise 10. You roll a die and toss a coin at the same time. If the coin lands on heads, you will double the value of the side showing the die. Otherwise, you will add 1 to the value of the side showing the die. What is the probability that you will get at least 5 when you perform this experiment?

Let's first calculate the number of total possibilities: $2\cdot 6$. For now, I'm going to ignore that there are lots of overlaps.

Suppose that the coin landed on heads, then what values on the die will give you a value of 4 or less? Only 1 and 2.

Suppose now that the coin landed on tails, then what calues on the die will give you a value of 4 of less? Now just 1, 2 and 3.

Those are then 5 possibilities out of 12. Therefore,

$$P(\leq 4)=\frac{5}{12}\Rightarrow P(\geq 5)=\frac{7}{12}$$

Exercise 11. You draw five cards from a standard shuffled deck of 52 cards. What is the probability that at least 3 of the cards you draw are diamonds?

Let's calculate the number of ways that all 5 are diamonds:

$$\binom{13}{5}$$

Now let's calculate the number of ways that exactly 4 are diamonds:

$$\binom{13}{4}\cdot 39$$

Now let's calculate the number of ways that exactly 3 are diamons:

$$\binom{13}{3}\binom{39}{2}$$

What is the total number of 5 card hands you can get?

$$\binom{52}{5}$$

The final answer:

$$\frac{\binom{13}{5} + \binom{13}{5} \cdot 39 + \binom{13}{3}\binom{39}{2}}{\binom{52}{5}}$$

Exercise 12. A dealer deals you a straight and your highest card is a seven. The dealer then proceeds to deal another player 5 cards. What is the probability that the player gets a straight that beats your straight?

How many hands can the other player have: $\binom{47}{5}$.

Let's assume that A's can be higher or low. Then there are then straights that start at A, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10. Only the straights that start at 4+ will beat you. If you've got a straight ending at 7, then you have a 4, 5, 6 and 7. Then if the straight starts at 4, there are:

$$3 \cdot 3 \cdot 3 \cdot 3 \cdot 4$$

available card combinations. If the straight starts at 5, then there are

$$3 \cdot 3 \cdot 3 \cdot 4 \cdot 4$$

available card combinations. If the straight starts at 6, then there are

$$3 \cdot 3 \cdot 4 \cdot 4 \cdot 4$$

available card combinations. If the straights starts at 7, then there are

$$3 \cdot 4 \cdot 4 \cdot 4 \cdot 4$$

available card combinations. If the straight starts at 8, 9 or 10 (3 possible), then there are 4^5 available card combinations.

Then we add together each of those cases to get:

$$\frac{3^4 \cdot 4 + 3^3 \cdot 4^2 + 3^2 \cdot 4^3 + 3^1 \cdot 4^4 + 3\left(4^5\right)}{\binom{47}{5}}$$

Exercise 13. A dealer deals you a flush and your highest card is a seven. The dealer then proceeds to deal another player 5 cards. What is the probability that the player gets a flush that beats your flush?

How many hands can the other player have: $\binom{47}{5}$.

Let's suppose that the player gets a flush in the suit you don't have (3 possible). If their high card is an 8, then they have $\binom{6}{4}$ choices for the other cards. If their high card is a 9, then they have $\binom{7}{4}$ choices for the other cards. You get the pattern, all the way up to A, where there are $\binom{12}{4}$ possible. In total, you've then got:

$$3\left[\sum_{k=6}^{12}\binom{k}{4}\right]$$

Now suppose that the player gets a flush in the suit you have. You have 5 of the 6 cards that are 7 or lower. That means, at minimum, their high card is a J (hand is 6,8,9,10,J). If their high card is a Q, then they have $\binom{6}{4}$ possible lower cards. If their high card is a K, they have $\binom{7}{4}$ and lastly, if their high card is an A, then they have $\binom{8}{4}$ possible. Adding this to the previous total, we get:

$$\frac{3\left[\sum_{k=6}^{12}\binom{k}{4}\right]+\sum_{k=6}^{8}\binom{k}{4}}{\binom{47}{5}}$$

Exercise 14. All the diamons have been removed from a deck of cards. You are dealt give cards. What is the probability that you are dealt a full house?

The number of possible five card hands that you're dealt is now:

$$\binom{39}{5}$$

Let's now choose the number that you're pairing up, you have 13 choices. Once you've chosen that, you've got to choose 2 suits, from 3 suits, for that pair of cards, $\binom{3}{2}$. Next, choose the 3 of a kind: 12 choices. Once you've chosen that, you have to select 3 suits, from 3 suits, for those 3 cards. Multiply those to get the final answer of:

$$\frac{13\cdot\binom{3}{2}\cdot 12\cdot\binom{3}{3}}{\binom{39}{5}}$$

Note that this disagrees with Dr. Weisbart's solution. Dr. Weisbart's solution uses $\binom{13}{2}$, which assumes that you can't tell the difference between which number is paired and which is a 3-of-a-kind.

Exercise 15. Suppose that you are dealt six cards from a fair shuffled deck. What is the probability that you get three pairs?

Total number of possible six card hands: $\binom{52}{6}$

Total number of three-pair hands. First choose the 3 numbers to pair up $\binom{13}{3}$. Then, for each pair you have, you have to choose 2 suits from 4. That gives you another $\binom{4}{2}^3$. Note that this excludes four of a kinds. After you select which number you have the pair in, you need to select the next card in the pair.

Final answer:

$$\frac{\binom{13}{3}\binom{4}{2}^3}{\binom{52}{6}}$$

Exercise 16. You are playing a game of blackjack. A dealer deals you a king and a seven from a standard shuffled deck. You decide to take a third card. What is the probability that you bust, that is, your new hand is worth more than 21 points?

What's the total number of cards that you could get: 50.

Of those cards, how many cards WOULD NOT bust you? You're only allowed 4 more points. Therefore, you could get 4 2's, 4 3's, 4 4's or 4 A's. That gives you 16 cards to NOT bust you, with a probability of $\frac{16}{50}$.

Therefore, the probability that you would bust is:

$$\frac{34}{50}$$

Exercise 17. You are playing a game of blackjack. A dealer deals you a king and a seven from a standard shuffled deck. You notice that three other players are playing and all of them have two cards showing with values greater than 8. You decide to take a third card. What is the probability that you bust, that is, you new hand is worth more than 21 points?

Let's think about what cards we could get to NOT bust. That doesn't change from before. BUT, the number of possible cards that you could get changes. There are 6 total cards out there that would bust you, but you can't get them anymore. That takes away from the total of 50 to give a total of 44.

The probability you won't bust is: $\frac{16}{44}$, so the probability you WOULD bust is:

$$\frac{28}{44}$$

Exercise 18. You draw five cards from a standard shuffled deck of 52 cards. What is the probability that you draw a straight flush?

Total number of 5 card hands? $\binom{52}{5}$.

Total number of straight flushes? Well, we've got 4 choices of suit. Then, we need to think of how many 5 card straights there are. If A's can be high or low, then we have a straight starting at A, 1, 2, 3, 4, 5, 6, 7, 8, 9 and 10. That gives us 10 possibilities. Multiply that by the number of suits we choose before to get the final answer of:

$$\frac{4 \cdot 10}{\binom{52}{5}}$$

Exercise 19. You draw five cards from a standard shuffled deck of 52 cards. What is the probability that you draw a straight flush if 2's are a wild card, that is, 2 can function as any card you wish.

The total number of 5 card hands doesn't change.

Let's choose the suit that we want to have the flush in, with 4 choices. Then, we think of how many 5 card straights there are irrespective of the special 2's. There are still 10. Now think of how many 4 card straights there are that don't include a 2. They start at 3, 4, 5, 6, 7, 8, 9, 10, J. There are 9 of those, times 4 choices of the wild 2. Now think of how many 3 card straights there are that don't include 2. They start at 3, 4, 5, 6, 7, 8, 9, 10, J, Q. There are 10 of these, times $\binom{4}{2}$ choices of the wild 2. Now think about how many 2 card straights there are: 11, times $\binom{4}{3}$ choices of the wild 2. Lastly, 1 card straights without 2: 12, times $\binom{4}{4}$ choices for the wild 2. Adding all of those cases up in the numerator, we get:

$$\frac{9 \cdot 4 + 10\binom{4}{2} + 11\binom{4}{3} + 12\binom{4}{4}}{\binom{52}{5}}$$

Exercise 20. You toss a biased coin two times. The probability for heads is $\frac{1}{4}$ and the probability for tails is $\frac{3}{4}$. What is the probability that the coin lands on heads and then lands tails? How many ways can this event occur?

Let's imagine this coin is a 4 sided dice, with sides being a tails. There are 16 total possibilities for ordered outcomes (some of which look the same). For the first flip, there's only 1 possibility to get heads first. There's then only 3 possibilities to get tails twice. Multiply those together and get:

$$\frac{3}{16} = \frac{1}{4} \cdot \frac{3}{4}$$

Exercise 21. You first roll a die. If the die lands on 1 or 2, you flip a fair coin twice. If the die lands on 3, 4, 5 or 6, you flip a coin one time. What is the probability that you toss a coin and it lands on heads? How many outcomes are possible for this event?

If you flip a fair coin twice, then you have a $\frac{3}{4}$ chance that you'll see a heads at least once. What's the probability that the die lands on 1 or 2: $\frac{1}{3}$.

Your final answer is:

$$\frac{1}{3}\frac{3}{4} + \frac{2}{3}\frac{1}{2} = \frac{1}{4} + \frac{1}{3} = \frac{7}{12}$$

The number of possible outcomes could get big fast (but it doesn't). There are 2 ways to get 1 or 2, then there are 3 possible outcomes. There are 4 ways to get 3+, then there are 2 possible outcomes. Multiplying within case and summing across case gives us:

$$2 \cdot 3 + 4 \cdot 2 = 6 + 8 = 14.$$

Exercise 22. Suppose that you have two 12 sided dice. You roll 2 dice. What is the probability of the number on the two dice is greater than 7? How many outcomes are possible for this event?

The probabiltiy that one dice has a value greater than 7 is $\frac{5}{12}$. Therefore, the probability that both are greater than 7 is $\left(\frac{5}{12}\right)^2$.

Now what if only one is greater than 7? If the first is greater than 7, but the second isn't, that happens with probability: $\frac{5 \cdot 7}{144}$. Now if the first is less than or equal to 7, but the second is greater, that happens with probability $\frac{7 \cdot 5}{144}$.

So the total probability is the sum of all those, or:

$$\frac{5^2 + 2 \cdot 5 \cdot 7}{144} = \frac{5^2 + 5 \cdot 7 + 7 \cdot 5}{144}$$

The total number of possible outcomes: $12 \cdot 12 = 144$. The number of possible desired outcomes: $144 \cdot \frac{5^2+2 \cdot 5 \cdot 7}{144} = 125$.

Exercise 23. Suppose you have two 12 sided dice. You roll two dice. What is the probability that the sum of the numbers on the two dice is less than 7? In how many ways can this occur?

There are still 144 possible outcomes.

How many dice pairs would have a sum less than 7? If the first is a 1, then there are 5 choices for the second. If the first is a 2, then there are 4 choices. Similarly for 3, there are 3 choices. And again for 4, there are 2 choices. Then if it's a 5, there is 1 choice. Let's multiply within case and add across cases:

$$1 + 2 + 3 + 4 + 5 = 15$$

The probability is then:

$$\frac{15}{144}$$