Math3C: Homework 4

by Wesley Kerr (TA: Section 3a and 3b)

You should NOT look at these answers before you attempt the problems. If you do, then you're not going to learn as well. Note that these answers are not reviewed by Dr. Weisbert and/or may include some errors (as we figured out first week). If you find one or more, let me know and I'll edit the document. My email is wesleytk@ucla.edu.

Exercise 1. You roll a fair six sided die. If you roll a 1, you pick a ball from bag 1. If you roll a 2, 3 or 4, you pick a ball from bag 2. If you roll a 5, you pick a ball from bag 3. If you roll a 6, you pick a ball from bag 4. Bag 1 has 2 R balls and 3 G balls. Bag 2 has 3 R balls and 4 G balls. Bag 3 has 1 R ball and 5 G balls. Bag 4 has 8 R balls and 2 G balls. What is the probability of picking a green ball given that you roll a 4?

This is just checking if you know what given means. You know you roll a 4, so you know that you're picking from bag 2. There are 4 green and 3 red balls in bag 2, so it's just:

$$P(G|4) = \frac{4}{7}$$

Exercise 2. A trick coin has a probability of $\frac{1}{3}$ of landing on heads and a probability of $\frac{2}{3}$ of landing on heads if it has just handed on heads, and a probability of $\frac{1}{4}$ of landing on heads if it has just landed on tails. What is the probability that the coin lands on heads on the first toss, tails on the second toss, and tails ont he third toss?

This is a big tree of probabilities:

$$P(HTT) = P(H) \cdot P(T|H \text{ last toss}) \cdot P(T|T \text{ last toss})$$

We know:

$$\begin{aligned} P(H|H \text{ last toss}) =& \frac{2}{3} \\ 1 =& P(H|H \text{ last toss}) + P(T|H \text{ last toss}) \\ P(T|H \text{ last toss}) =& 1 - \frac{2}{3} = \frac{1}{3} \end{aligned}$$

A similar thing can be done to show that:

$$P(T|T \text{ last toss}) = 1 - P(H|T \text{ last toss}) = 1 - \frac{1}{4} = \frac{3}{4}$$

Plugging those numbers in, we get:

$$P(HTT) = \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{3}{4}$$

Exercise 3. A trick coin has a probability of $\frac{1}{3}$ of landing on heads and a probability of $\frac{2}{3}$ of landing on heads if it has just landed on heads and a probability of $\frac{1}{4}$ of landing on heads if it has just landed on tails. What is the probability that the coin lands on heads on the first toss and tails on the second toss?

Wait, we just did that. It's just the first two tosses of the last exercise:

$$P(HT) = \frac{1}{3} \cdot \frac{1}{3}$$

Exercise 4. Suppose that you are dealt a five hard hand from a shuffled 52 card deck. Suppose that the dealer tell you that you have at least 4 diamongs. What is the probability that you are holding a flush? Compare this with the probability that you get a flush if you are dealth four diamongs, then a fifth card.

Let's do the easy thing first: the second question. If you already have 4 cards that are diamonds, then you have $52 - 4 = 48$ cards left in the deck, $13 - 4 = 9$ of which are diamonds. That means that you have:

$$P(\text{flush — 1st 4 are diamonds}) = \frac{9}{48}$$

Now the harder one.

$$\begin{aligned} P(\text{flush} \mid \geq 4 \text{ diamonds}) =& \frac{\#\text{flush} \cap \geq 4\text{diamonds}}{\# \geq 4\text{diamonds}} \\ =& \frac{\binom{13}{5}}{P(\text{ 4 diamonds}) + P(\text{5 diamonds})} \\ =& \frac{\binom{13}{5}}{\binom{13}{4}\binom{39}{1} + \binom{13}{5}} \end{aligned}$$

It's 13 choose 5 because there are 13 diamonds, of which we need 5 in our hand to get a flush. For the other term, we choose 4 diamonds to have from 13, then we choose the last card (which is not a diamond [52-13=39]) from the 39 remaining cards.

Exercise 5. Suppose that you are dealt a five card hand from a shuffled 52 card deck. Suppose that the dealer tells you that you have at least 3 diamonds. What is the probability that you are holding a flush?

Yay repeating the same work as the last problem. This time, you're got another term where you have 3 diamonds, so you choose 3 from 13 diamonds, then 2 from the 39 remaining cards. Writing the full thing out:

$$P(\text{flush — } \geq 3 \text{ diamonds}) = \frac{\binom{13}{5}}{\binom{13}{3}\binom{39}{2} + \binom{13}{4}\binom{39}{1} + \binom{13}{5}}$$

Exercise 6. You roll a fair six sided die. If you roll a 1, you pick a ball from bag 1. If you roll a 2, 3 or 4, you pick a ball from bag 2. If you roll a 5, you pick a ball from bag 3. If you roll a 6, you pick a ball from bag 4. Bag 1 has 2 R balls and 3 G balls. Bag 2 has 3 R balls and 4 G balls. Bag 3 has 1 R ball and 5 G balls. Bag 4 has 8 R balls and 2 G balls. What is the probability that you pick a red ball?

This is law of total probability!

$$\begin{aligned} P(R) =& P(R|1)P(1) + P(R|2,3,4)P(2,3,4) + P(R|5)P(5) + P(R|6)P(6) \\ =& \frac{2}{5} \cdot \frac{1}{6} + \frac{3}{7} \cdot \frac{3}{6} + \frac{1}{6} \cdot \frac{1}{6} + \frac{8}{10} \cdot \frac{1}{6} \end{aligned}$$

Exercise 7. You roll a fair six sided die. If you roll a 1, you pick a ball from bag 1. If you roll a 2, 3 or 4, you pick a ball from bag 2. If you roll a 5, you pick a ball from bag 3. If you roll a 6, you pick a ball from bag 4. Bag 1 has 2 R balls and 3 G balls. Bag 2 has 3 R balls and 4 G balls. Bag 3 has 1 R ball and 5 G balls. Bag 4 has 8 R balls and 2 G balls. What is the probability that you pick a green ball?

This is law of total probability!

$$\begin{aligned} P(G) =& P(G|1)P(1) + P(G|2,3,4)P(2,3,4) + P(G|5)P(5) + P(G|6)P(6) \\ =& \frac{3}{5} \cdot \frac{1}{6} + \frac{4}{7} \cdot \frac{3}{6} + \frac{5}{6} \cdot \frac{1}{6} + \frac{2}{10} \cdot \frac{1}{6} \end{aligned}$$

Note that Weisbart has a clear typo of $\frac{7}{6}$ in his solution. Probabilities can never be greater than 1.

Exercise 8. You roll a fair six sided die. If you roll a 1, you pick a ball from bag 1. If you roll a 2, 3 or 4, you pick a ball from bag 2. If you roll a 5, you pick a ball from bag 3. If you roll a 6, you pick a ball from bag 4. Bag 1 has 2 R balls and 3 G balls. Bag 2 has 3 R balls and 4 G balls. Bag 3 has 1 R ball and 5 G balls. Bag 4 has 8 R balls and 2 G balls. What is the probability that you pick a green ball given that you roll a number greater than 4?

Remember that:

$$P(G| > 4) = P(G|5 \cup 6) = \frac{P(G \cap (5 \cup 6))}{P(5 \cup 6)} = \frac{P(G|5)P(5) + P(G|6)P(6)}{P(5) + P(6)} = \frac{\frac{5}{6}\frac{1}{6} + \frac{2}{10}\frac{1}{6}}{\frac{2}{6}}$$

Exercise 9. Suppose that Team A has a 10% chance of playing Team B and a 90% chance of playing Team C in the next meet. Team A has a 70% chance of beating Team B and a 40% chance of beating Team C. What is the probability that Team A wins the next game?

Yay more law of total probability.

$$\begin{aligned} P(\text{A wins}) =& P(\text{A wins}|\text{plays B})P(\text{plays B}) + P(\text{A wins}|\text{plays C})P(\text{plays C}) \\ =& 0.7 \cdot 0.1 + 0.4 \cdot 0.9 \end{aligned}$$

Exercise 10. Suppose that you take three cards from a fair deck. If the first is a Jack and the second is a diamond, what is the probability that the third is the queen of hearts?

You've removed two cards from the deck. Now you want one specific card of those 52-2=50. This is just:

$$\frac{1}{50}$$