Math3C: Homework 7

by Wesley Kerr (TA: Section 3a and 3b)

You should NOT look at these answers before you attempt the problems. If you do, then you're not going to learn as well. Note that these answers are not reviewed by Dr. Weisbert and/or may include some errors (as we figured out first week). If you find one or more, let me know and I'll edit the document. My email is wesleytk@ucla.edu.

Exercise 1. Find c so that

$$\rho(x) = \begin{cases} cx^3 & \text{if } x \in [1,2] \\ 0 & \text{otherwise} \end{cases}$$

is a probability density function. If X is a random variable with probability density function $\rho(x)$, then calculate $E(X)$ and $Var(X)$.

First, we know that:

$$\begin{aligned} \int_1^2 cx^3 dx &= 1 \\ c\left.\frac{x^4}{4}\right|_{x=1}^{x=2} &= 1 \\ c\left[\frac{2^4}{4} - \frac{1}{4}\right] &= 1 \\ c &= \frac{1}{4 - \frac{1}{4}} \\ &= \frac{4}{16-1} \\ &= \frac{4}{15} \end{aligned}$$

Then we know that:

$$\begin{aligned} E(X) &= \int_1^2 x \cdot \frac{4}{15} x^3 dx \\ &= \frac{4}{15}\int_1^2 x^4 dx \\ &= \frac{4}{15}\left[\left.\frac{x^5}{5}\right|_{x=1}^{x=2}\right] \\ &= \frac{4}{15}\left[\frac{2^5}{5} - \frac{1}{5}\right] \end{aligned}$$

We also know that:

$$Var(X) = E(X^2) - E(X)^2$$

Let's solve for that:

$$\begin{aligned}
E(X^2) &= \int_1^2 x^2 \cdot \frac{4}{15}x^3 dx \\
&= \frac{4}{15}\int_1^2 x^5 dx \\
&= \frac{4}{15}\left[\left.\frac{x^6}{6}\right|_{x=1}^{x=2}\right] \\
&= \frac{4}{15}\left[\frac{2^6}{6} - \frac{1}{6}\right]
\end{aligned}$$

Then we plug in:

$$Var(X) = \frac{4}{15}\left[\frac{2^6}{6} - \frac{1}{6}\right] - \left(\frac{4}{15}\left[\frac{2^5}{5} - \frac{1}{5}\right]\right)^2$$

Exercise 2. Find c so that

$$\rho(x) = \begin{cases} 5x & \text{if } x \in [-1, c] \\ 0 & \text{otherwise} \end{cases}$$

is a probability density function.

This is the same as the first part:

$$\begin{aligned}
\int_{-1}^{c} 5x dx &= 1 \\
5\left.\frac{x^2}{2}\right|_{x=-1}^{x=c} &= 1 \\
5\left[\frac{c^2}{2} - \frac{1}{2}\right] &= 1 \\
\frac{c^2}{2} - \frac{1}{2} &= \frac{1}{5} \\
c^2 &= 2 \cdot \left[\frac{1}{5} + \frac{1}{2}\right] \\
c &= \sqrt{2 \cdot \left[\frac{1}{5} + \frac{1}{2}\right]}
\end{aligned}$$

BUT, $\rho(x) < 0$ if $x < 0$ and probability densities are never negative, so there is no value for c that can make this a probability density.

Exercise 3. Let X and Y be random variables such that the PDF for X is ρ_X and the PDF for Y is ρ_Y. Suppose that

$$\rho_X(x) \begin{cases} \frac{1}{2} & \text{if } x \in [0, 2] \\ 0 & \text{otherwise} \end{cases} \quad \text{and } \rho_Y(x) \begin{cases} 3x^2 & \text{if } x \in [0, 1] \\ 0 & \text{otherwise} \end{cases}$$

Let $Z = \max(X, Y)$. Calculate $E[Z]$.

Assuming independence, we say: the probability distribution for the maximum is:

$$\begin{aligned}
\rho_z =& P(X = z)P(Y \leq z) + P(X < z)P(Y = z) \\
=& \begin{cases} \frac{1}{2} \cdot 1 + 0 \cdot \int_0^z \frac{1}{2} dz & \text{if } z \in [1,2] \\ \frac{1}{2} \int_0^z 3z^2 dz + 3z^2 \int_0^z \frac{1}{2} dz & \text{if } z \in [0,1] \\ 0 & \text{otherwise} \end{cases} \\
=& \begin{cases} \frac{1}{2} & \text{if } z \in [1,2] \\ \frac{1}{2}\frac{3z^3}{3} + 3z^2 \frac{z}{2} & \text{if } z \in [0,1] \\ 0 & \text{otherwise} \end{cases} \\
=& \begin{cases} \frac{1}{2} & \text{if } z \in [1,2] \\ \frac{z^3}{2} + \frac{3z^3}{2} & \text{if } z \in [0,1] \\ 0 & \text{otherwise} \end{cases} \\
=& \begin{cases} \frac{1}{2} & \text{if } z \in [1,2] \\ \frac{4z^3}{2} & \text{if } z \in [0,1] \\ 0 & \text{otherwise} \end{cases} \\
=& \begin{cases} \frac{1}{2} & \text{if } z \in [1,2] \\ 2z^3 & \text{if } z \in [0,1] \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

Then $E(Z)$ is just the integral of that, or:

$$\begin{aligned}
E(Z) =& \int_0^1 z \cdot 2z^3 dz + \int_1^2 z \cdot \frac{1}{2} dz \\
=& \int_0^1 2z^4 dz + \int_1^2 \frac{z}{2} dz \\
=& 2\frac{1}{5} + \frac{1}{2}\left[\frac{2^2}{2} - \frac{1}{2}\right] \\
=& \frac{2}{5} + \frac{1}{2}\frac{3}{2} \\
=& \frac{2}{5} + \frac{3}{4}
\end{aligned}$$

Exercise 4. Let X and Y be random variables such that the PDF for X is ρ_X and the PDF for Y is ρ_Y. Suppose that:

$$\rho_X(x) \begin{cases} 3e^{-3x} & \text{if } x \in [0,\infty) \\ 0 & \text{otherwise} \end{cases} \quad \text{and } \rho_Y(x) \begin{cases} 2e^{-2x} & \text{if } x \in [0,\infty) \\ 0 & \text{otherwise} \end{cases}$$

Let $Z = \max(X, Y)$. Calculate $E[Z]$.

We do this the same as the last problem, only the integrals are harder.

$$\begin{aligned}
\rho_z =& P(X = z)P(Y \leq z) + P(X < z)P(Y = z) \\
=& \begin{cases} 3e^{-3z}\int_0^z 2e^{-2z}dz + 2e^{-2z}\int_0^z 3e^{-3z}dz & \text{if } z \in [0,\infty) \\ 0 & \text{otherwise} \end{cases} \\
=& \begin{cases} 3e^{-3z}2\frac{1-e^{-2z}}{2} + 2e^{-2z}\frac{1-e^{-3z}}{3}3 & \text{if } z \in [0,\infty) \\ 0 & \text{otherwise} \end{cases} \\
=& \begin{cases} 3e^{-3z}\left(1-e^{-2z}\right) + 2e^{-2z}\left(1-e^{-3z}\right) & \text{if } z \in [0,\infty) \\ 0 & \text{otherwise} \end{cases} \\
=& \begin{cases} 3e^{-3z} - 3e^{-3z}e^{-2z} + 2e^{-2z} - 2e^{-2z}e^{-3z} & \text{if } z \in [0,\infty) \\ 0 & \text{otherwise} \end{cases} \\
=& \begin{cases} 3e^{-3z} + 2e^{-2z} - 5e^{-5z} & \text{if } z \in [0,\infty) \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

Now we just need to calculate the expectation:

$$E(Z) = \int_0^\infty z\left(3e^{-3z} + 2e^{-2z} - 5e^{-5z}\right)dz = \quad 3\int_0^\infty ze^{-3z}dz + 2\int_0^\infty ze^{-2z}dz - 5\int_0^\infty ze^{-5z}dz$$

This should be done using integration by parts, or knowing that:

$$\int_0^\infty xe^{-kx}dx = \frac{1}{k^2}$$

Plugging that in:

$$E(Z) = 3\frac{1}{3^2} + 2\frac{1}{2^2} - 5\frac{1}{5^2} = \frac{1}{3} + \frac{1}{2} - \frac{1}{5}$$

Exercise 5. Suppose that you a radioactive atom that has a half-life of 100 days. What is the probability that it decays within 120 days? What is the probability that it decays within 200 days? What is the expected length of time for the atom to decay?

This is just memorization of the exponential formulae. Let's first figure out the half life:

$$\begin{aligned}
P(X > t_{1/2}) = 0.5 =& e^{-\lambda t_{1/2}} \\
\ln(0.5) =& -\lambda t_{1/2} \\
\frac{-\ln(0.5)}{t_{1/2}} =& \lambda \\
\frac{-\ln(0.5)}{100} =& \lambda
\end{aligned}$$

Now the probability that it decays within 120 days?

$$\begin{aligned}
P(X \leq 120) =& 1 - P(X > 120) \\
=& 1 - e^{-\lambda t} \\
=& 1 - e^{-\frac{-\ln(0.5)}{100}120} = 1 - \left(\frac{1}{2}\right)^{\frac{120}{100}}
\end{aligned}$$

Now the probability that it decays within 200 days?

$$P(X \leq 20) = e^{-\frac{-\ln(0.5)}{100}200} = 1 - \left(\frac{1}{2}\right)^{\frac{200}{100}} = 1 - \left(\frac{1}{2}\right)^2 = 1 - \frac{1}{4}$$

What is the expected length of time for the atom to decay? This is just the expectation formula:

$$E(X) = \frac{1}{\lambda} = \frac{100}{-\ln(0.5)} = \frac{100}{\ln 2}$$

Exercise 6. Suppose that you have 10 radioactive atoms of a substance that has a half-life of 2 years. What is the probability that at least 2 atoms decay in 4 years?

You could do this using the formulae, or you could recognize that 4 years is twice the half life. Half the atoms decay for each half life, therefore:

$$P(X \leq 4) = P(X \leq 2)^2 = \frac{1}{4}$$

That's the probability of being left. That means the probability of decay is $\frac{3}{4}$. Now we have 10 atoms. This is then: $P(N \geq 2)$ where N is a binomial distribution with $n = 10$ and $p = \frac{1}{4}$.

$$\begin{aligned} P(N \geq 2) =& 1 - P(N = 0) - P(N = 1) \\ =& 1 - \left[\frac{1}{4}\right]^{10} - \binom{10}{1}\left[\frac{1}{4}\right]^{9}\left[\frac{3}{4}\right] \end{aligned}$$

For $N = 0$, we want zero particiles to decay. That means that 10 atoms are left at 4 years.

Exercise 7. The probability that a certain atom decays in 10 days is 35%. What is the half-life of the atom?

Yay more algebra. Using the formula:

$$\begin{aligned} P(X \leq 10) = 0.35 =& 1 - e^{-\lambda \cdot 10} \\ 0.65 =& e^{-\lambda \cdot 10} \\ \ln 0.65 =& -\lambda \cdot 10 \\ \frac{-\ln 0.65}{10} =& \lambda \end{aligned}$$

The half life is:

$$\begin{aligned} P(X \leq t_{1/2}) = 0.5 =& e^{-\lambda t_{1/2}} \\ \frac{-\ln(0.5)}{t_{1/2}} =& \lambda \\ \frac{\ln 2}{\lambda} =& t_{1/2} \\ \ln 2 \frac{10}{-\ln 0.65} =& t_{1/2} \end{aligned}$$

Exercise 8. Suppose that you have two radioactive atoms. One atom has a half-life of five hours and the other atom has a half-life of eight hours. What is the expected amount of time you must wait until the first decay event?

You can do this two ways: the long way (see Weisbart's solution), and the short way. The short way is to realize that the time to wait until the first decay is just a process with rate of:

$$\lambda_{first} = \lambda_1 + \lambda_2$$

We have from before that

$$\frac{\ln 2}{t_{1/2}} = \lambda$$

That means that:

$$\begin{aligned}
\frac{\ln 2}{t_{1/2,first}} &= \frac{\ln 2}{t_{1/2,1}} + \frac{\ln 2}{t_{1/2,2}} \\
t_{1/2,first} &= \left(\frac{1}{t_{1/2,1}} + \frac{1}{t_{1/2,2}}\right)^{-1} \\
&= \frac{t_{1/2,1} t_{1/2,2}}{t_{1/2,1} + t_{1/2,2}} = \frac{5 \cdot 8}{5+8} = \frac{40}{13}
\end{aligned}$$

Now we know that:

$$\lambda = \frac{\ln 2}{t_{1/2,first}} = \ln 2 \frac{13}{40}$$

And lastly:

$$E(Z) = \frac{1}{\lambda} = \frac{40}{13 \ln 2}$$

Exercise 9. Suppose that you have two radioactive atoms. One atom has a half-life of five hours and the other atom has a half-life of eight hours. What is the expected amount of time you must wait until both atoms decay?

There's no easy way to do this (until you do it the long way). We need to figure out:

$$P(Z = \max\{X, Y\} = z)$$

where X is the first atom and Y is the second atom. From before, we know:

$$\begin{aligned}
P(Z = z) &= P(X = z)P(Y \leq z) + P(X < z)P(Y = z) \\
&= \lambda_X e^{-\lambda_X z}\left(1 - e^{-\lambda_Y z}\right) + \left(1 - e^{-\lambda_X z}\right)\lambda_Y e^{-\lambda_Y z} \\
&= \lambda_X e^{-\lambda_X z} + \lambda_Y e^{-\lambda_Y z} - (\lambda_X + \lambda_Y)\, e^{-(\lambda_X + \lambda_Y) z}
\end{aligned}$$

Hold on. We recognize that! It's the sum of three exponential processes (with a negative on the third one). We then know that:

$$E(Z) = \int_0^\infty z P(Z = z) dz = \int_0^\infty z P(X = z) dz + \int_0^\infty z P(Y = z) dz - \int_0^\infty z P(Y + X = z) dz$$

$$E(Z) = \frac{1}{\lambda_X} + \frac{1}{\lambda_Y} - \frac{1}{\lambda_X + \lambda_Y}$$

Now we can use:

$$\lambda = \frac{\ln 2}{t_{1/2}}$$

Plugging that in:

$$\begin{aligned}
E(Z) &= \frac{5}{\ln 2} + \frac{8}{\ln 2} - \frac{1}{\frac{\ln 2}{5} + \frac{\ln 2}{8}} \\
&= \frac{5}{\ln 2} + \frac{8}{\ln 2} - \frac{5 \cdot 8}{\ln 2 (5+8)} \\
&= \frac{13}{\ln 2} - \frac{40}{13 \ln 2}
\end{aligned}$$

Exercise 10. A certain substance has a half-life of 20 days. You have 10^9 atoms of the substance. Find a lower bound on the probability that between 49.5% and 50.5% of the atoms have decayed in 20 days.

You've got a range, so Chebychev screams to you from your head. To remind you, Chebychev says:

$$P(|X - E(X)| \geq c) \leq \frac{Var(X)}{c^2}$$

If X is a binomial proportion of success, then we have:

$$P(|X - E(X)| \geq c) \leq \frac{1}{4nc^2}$$

We remember that:

$$P(|X - E(X)| \geq c) = 1 - P(|X - E(X)| < c)$$

Plugging that in:

$$\begin{aligned} 1 - P(|X - E(X)| < c) \leq& \frac{1}{4nc^2} \\ -P(|X - E(X)| < c) \leq& \frac{1}{4nc^2} - 1 \\ P(|X - E(X)| < c) \geq& 1 - \frac{1}{4nc^2} \end{aligned}$$

We know $n = 10^9$ and $c = 0.5\%$, so we plug in:

$$P(|X - E(X)| < c) \geq 1 - \frac{1}{4nc^2} = 1 - \frac{1}{4 \cdot 10^9 \cdot (0.005)^2} = 0.99999$$

Not the that we assumed that $E(X) = 50\%$. This is true because you were asking about the number of atoms after a half life. Otherwise, we wouldn't use Chebychev.