Math3C: Things to Memorize for Final

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This list was selected without conferring with Dr. Weisbart or seeing the final exam. If you think something is missing, please let me know and I'll add it, if it's critical (wesleytk@ucla.edu).

Counting

Cases vs. tuples: Cases Add. Tuples Multiply.

- Cases: Different options, where at the end of the day you only have one or other other.
- Tuples: Different steps/stages of a problem, where for each choice you make in one step, you can make every possible choice in the next step.

The number of ways to order n distinguishable objects: $n!$.

The number of ways to order n objects, when k are identical:

$$\frac{n!}{k!}$$

When you care about order (they are distinguishable) for k objects from n, but you don't care about the order for the remaining $n-k$.

$$P(n,k) = \frac{n!}{k!}$$

When you don't care about order, and have 2 types of indistinguishable objects. This is also the coefficient to a binomial expansion ($(x+y)^n$) for the term with k x's or y's.

$$C(n,k) = \frac{n!}{k!(n-k)!}$$

When you're got a certain number of types of objects (types are distinguishable), k, (like fingers or soda types), and a certain number (counts are indistiguishable), n, of them that you can bring. You've got:

$$\binom{n+k-1}{n} = \binom{n+k-1}{k-1}$$

Common mistake that will get you 0 points: using $\binom{n+k-1}{k}$.

Probability

Definition of probability:

$$P = \frac{\#\text{Successes}}{\#\text{Total Outcomes}}$$

Law of total probabilty: If only A and B occur, then:

$$P(C) = P(C|A)P(A) + P(C|B)P(B)$$

Classic problems are where A and B are the outcome of dice rolls or coin tosses and C is selecting balls from different bags, depending on the outcome of the dice roll or coin toss. Alternatively, A and B are the presence or absence of disease and C is the outcome of a diagnostic test.

Conditional Probabilities:

$$\begin{aligned}
P(A \cap B) =& P(A|B)P(B) \\
P(A \cup B) =& P(A) + P(B) - P(A \cup B) \\
P(A|B) =& \frac{P(A \cap B)}{P(B)} = \frac{P(B|A)P(A)}{P(B)} \\
E(X) =& E(X|A)P(A) + E(X|B)P(B)
\end{aligned}$$

Two events, A and B are independent if and only if:

$$P(A \cap B) = P(A)P(B) \text{ and } P(A|B) = P(A) \text{ and } P(B|A) = P(B)$$

If you show one of those, you've shown them all.

Bayes Rule: This is listed above, but let's write it out specifically:

$$P(D|-) = \frac{P(-|D)P(D)}{P(-)} = \frac{P(-|D)P(D)}{P(-|D)P(D) + P(-|D^c)P(D^c)}$$

The only other trick is to know that

$$\begin{aligned}
P(-|D) =& \text{False Negatives} \\
P(+|D) =& 1 - P(-|D) = \text{True Positives} \\
P(+|D^c) =& \text{False Positives} \\
P(-|D^c) =& 1 - P(+|D^c) = \text{True Negatives}
\end{aligned}$$

Just because this comes up a lot, if you've got a sequence of independent tests:

$$P(D| - +) = \frac{P(-|D)P(+|D)P(D)}{P(-|D)P(+|D)P(D) + P(-|D^c)P(+|D^c)P(D^c)}$$

General properties of discrete and continuous random variables and their expectations/variances:

$$\begin{aligned}
1 =& \sum_x P(X = x) = \int_{-\infty}^{\infty} \rho_X dx \\
E(X) =& \sum_x xP(X = x) = \int_{-\infty}^{\infty} x\rho_X dx \\
E(aX + b) =& aE(X) + b \\
E(X^2) =& \sum_x x^2 P(X = x) = \int_{-\infty}^{\infty} x^2 \rho_X dx \\
Var(X) =& \sum_x [X - E(X)]^2 P(X = x) = E(X^2) - [E(X)]^2 \\
Var(aX + b) =& a^2 Var(X) \\
Var\left(\frac{1}{n}\sum_{k=1}^{n} X\right) =& \frac{Var(X)}{n} \text{ aka weak law of large numbers}
\end{aligned}$$

Binomial Distribution: Set number of trials, n, that are all independent and have the same probability of occuring, p. Questions will ask for the probability you see a certain number of successes, k.

$$\begin{aligned}
P(X=k|n,p) =& \binom{n}{k} p^k (1-p)^{n-k} \\
E(X) =& np \\
Var(X) =& np(1-p) \\
2^n =& \sum_{k=0}^{n} \binom{n}{k}
\end{aligned}$$

Geometric Distribution: Set number of successes, $k = 1$. Questions will ask for the number of trials till you get that success.

$$\begin{aligned}
P(X=n|p) =& (1-p)^{k-1} p \\
E(X) =& \frac{1}{p} \\
\sum_{k=0}^{\infty} r^k =& \frac{1}{1-r}
\end{aligned}$$

Exponential Distribution: This is the continuous analog of the Geometric. The classic problem relates to decay of radioactive particles.

$$\begin{aligned}
P(T > t) =& e^{-\lambda t} \\
P(T \leq t) =& 1 - e^{-\lambda t} \\
\rho_T =& \lambda e^{-\lambda t} \\
E(T) =& \frac{1}{\lambda} \\
\lambda =& \frac{\ln 2}{t_{1/2}}
\end{aligned}$$

Inequalities If X and a are always ≥ 0 then:

$$\begin{aligned}
P(X \geq a) \leq& \frac{E(X)}{a} \\
P(|X - E(X)| \geq c) \leq& \frac{Var(X)}{c^2} \\
P(|X - E(X)| < c) \geq& 1 - \frac{Var(X)}{c^2}
\end{aligned}$$

ONLY IF X is a proportion and c also is a proportion, you can simplify this to

$$\begin{aligned}
P(|X - E(X)| \geq c) \leq& \frac{1}{4nc^2} \\
P(|X - E(X)| < c) \geq& \frac{1}{4nc^2}
\end{aligned}$$

Integrals

$$\begin{aligned}
\int x^k dx =& \frac{x^{k+1}}{k+1} \\
\int e^{-\lambda t} dt =& \frac{e^{-\lambda t}}{-\lambda} \\
e^{-\lambda \infty} =& 0 \\
e^{-\lambda \cdot 0} =& 1 \\
\int_0^\infty t e^{-\lambda t} dt =& \frac{1}{\lambda^2} \\
\int t e^{-\lambda t} dt =& \frac{t e^{-\lambda t}}{-\lambda} - \frac{e^{-\lambda t}}{\lambda^2}
\end{aligned}$$

Remember integration by parts? Let's use the last question as an example:

$$\begin{aligned}
u =& t \\
du =& dt \\
dv =& e^{-\lambda t} dt \\
v =& \frac{e^{-\lambda t}}{-\lambda} \\
\int t e^{-\lambda t} dt =& uv - \int v du \\
=& \frac{t e^{-\lambda t}}{-\lambda} - \int \frac{e^{-\lambda t}}{-\lambda} dt \\
=& \frac{t e^{-\lambda t}}{-\lambda} - \frac{e^{-\lambda t}}{\lambda^2}
\end{aligned}$$