Math3C: Midterm
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Note that these answers are not reviewed by Dr. Weisbert and/or may include some errors. If you find one or more, let me know and I'll edit the document. My email is wesleytk@ucla.edu.

C Level Questions

C1. Four friends go to a candy shop that has six types of candy. Each friend buys one piece of candy. How many ways can the friends get candy if they simultaneously make their purchase?

How many friends are in our final set: 4. They are all distinguishable. How many choices do each of them have: 6.

$$6^4$$

C2. How many ways can you select a team of eight people from a group of 30 if you do not care about the order in which the people are selected?

This is just $\binom{30}{8}$. This is the definition of binomial choice.

C3. How many words (nonsense words included) can you make using all of the letters in the word XXYYYYZZZWWWWW?

The word is 14 long, with 2 X's, 4 Y's, 3 Z's and 5 W's. That gives us:

$$\frac{14!}{2!4!3!5!}$$

Remember if they were all distinguishable, then we'd have 14!, but we divide by the number of ways we can reorder each of the indistinguishable objects.

C4. How many ways can you make a bouquet of 12 flowers if there are 5 types of flowers from which to select the 12?

Key word: types! That's a rings and fingers problem with $k = 5$ and $n = 12$. We've got:

$$\binom{n+k-1}{k-1} = \binom{12+5-1}{5-1}$$

C5. Six students will be in the math club and eight students will be in the physics club. In how many ways can the groups be formed if members can serve in only one group at a time and the students are chosen from a class of 30?

The two clubs are distinguishable (becuase they're labelled), so let's choose math first because we love math. We've got $\binom{30}{6}$ choices. From the remaining kids, we choose the second tier club, physics, with $\binom{30-6}{8}$ choices. For every math club choice, we've got that many choices of physics clubs, so we multiply:

$$\binom{30}{6}\binom{24}{8}$$

C6. You roll a fair six sided die two times. What is the probability that the sum of the two rolls is 3?

How many ways can we get that sum: 1+2 and 2+1. How many ways can two six sided die come out: $6 \cdot 6$. We have:

$$P = \frac{\#wins}{\#total} = \frac{2}{36}$$

C7. You have a fair coin, a fair eight sided die, and a fair 10 sided die. You flip the coin. If the coin lands on heads, you roll the eight sided die. If the coin lands on tails, you roll the 10 sided die. What is the probability that you roll a 4 given the coin lands on heads?

The given statement lets you ignore a lot of the question. You actually only care about "If the coin lands on heads, you roll the eight sided die." because you know the coin landed on heads. Therefore, the probability is:

$$\frac{1}{8}$$

C8. You have a fair coin, a fair eight sided die, and a fair 10 sided die. You flip the coin. If the coin lands on heads, you roll an eight sided die. If the coin lands on tails, you roll the 10 sided die. What is the probability that you roll a 2?

Now we use the law of total probability:

$$P(2) = P(2|H)P(H) + P(2|T)P(T) = \frac{1}{8}\frac{1}{2} + \frac{1}{10}\frac{1}{2}$$

B Level Questions

B1. You have six shirts, two pairs of pants, three hats, and five identical silver rings. You will wear one of each of the shirts, pants, and has and you will wear all five silver rings on your fingers. How many outfits can you make? Note: Rings on different fingers will count as different outfits.

First of all, you're not going to wear a shirt on you head, so there's no interaction between clothing types. Each clothing type is a sub problem. For each choice of shirt, we can switch all the rest of the clothing, so they should multiply. That means we've got:

$$shirts \cdot pants \cdot hats \cdot rings = \binom{6}{1}\binom{2}{1}\binom{3}{1}\binom{5+10-1}{10-1} = \binom{6}{1}\binom{2}{1}\binom{3}{1}\binom{5+10-1}{5}$$

B2. You seat seven people at a round table. Alice and Bob refuse to sit apart since they are married. You then take a photograph of the table from above and have everyone look up (the print of the photo is round so you cannot distinguish the chairs). How many photos can you make?

First of all, Alice and Bob count as one person. Let's call them Aob. Then you've got 6 people sitting in a round table. If they were in a line, you'd have 6! option. But since they're in a circle, there are 6 rotations of that line that are all identical. Now let's think about Aob. They can also switch places, giving us 2! more choices. Final answer:

$$2! \cdot \frac{6!}{6} = 5! \cdot 2!$$

B3. Ten physicists, eight mathematicians, and 12 engineers are at a conference. Three of the physicists, four of the mathematicians and two of the engineers are selected to take part in a group photo. How many photographs are possible? Assume that all people look different.

Let's first choose the people: $\binom{10}{3}\binom{8}{4}\binom{12}{2}$. Now that we've got 9 people, we have 9! ways to order them. For every selection of people, we've got new ways to order, so we've got:

$$\binom{10}{3}\binom{8}{4}\binom{12}{2}9!$$

B4. You are to make a bouquet of 15 floers. There are five types of flowers from which to select the 15. The bouquet must contain at least four roses and there are six possible vases to use for the bouquet. How many different bouquets can you make (the same flowers in different vases will count as different bouquets)?

At the end of the day, we're going to just have one vase, and one set of flowers. That means the final choice of bouquet multiplies our answer by 6 and that's it. I interpret the problem this way because otherwise, we're going to have a huge number of cases and this is just a B level problem.

Now let's chose the flowers. We know we have at least 4 roses, so we're really chosing 11 flowers. We can still choose roses, so we've got 5 types. Rings and fingers problem. That means our final answer is:

$$6 \cdot \binom{11+5-1}{11}$$

B5. A group of 20 people meet at the park for a pick-up basketball game. Each team will consist of five players. On the board: 2 teams.

Let's assume that you're making a yellow and a blue team (because we're at UCLA). We'd then have $\binom{20}{5}$ choices for the yellow team and the remaining team would have $\binom{15}{5}$ choices. BUT, we didn't label the teams, so we have to remove the distinction of yellow and blue. That's like removing order in a 2 letter word. Final answer:

$$\binom{20}{5}\binom{15}{5}\frac{1}{2!}$$

B6. You have a bag with four red balls and five green balls. You randomly take three balls from the bag. What is the probability that you take at least 2 red balls?

There is an easy way and a hard way. Let's do the easy way:

$$\begin{aligned} 1 =& P(0R) + P(1R) + P(\geq 2R) \\ P(\geq 2R) =& 1 - P(0R) + P(1R) \end{aligned}$$

The probability of no red balls is:

$$\binom{5}{3}$$

Because you have successively less green balls left to choose. That leaves you with GGG. Now the probability of one red ball is:

$$\binom{5}{2}\binom{4}{1}$$

The total number of choices of balls is $\binom{9}{3}$. Final answer:

$$1 - \frac{\binom{5}{3} + \binom{5}{2}\binom{4}{1}}{\binom{9}{3}}$$

B7. You have two coins in your pocket, a two headed half dollar and standard half dollar. You randomly pick a coin and toss it. It lands on heads. What is the probability that it is a standard coin?

We have $P(S|H) = \frac{P(H \cap S)}{P(H)}$. Doing the tree, we have $P(H) = \frac{1}{2} + \frac{1}{2}\frac{1}{2}$ because every time you toss the two headed, you'll get a head and half the time you toss the fair coin, you'll get a head. That latter term is also $P(H \cap S) = \frac{1}{2}\frac{1}{2}$. Dividing those gives us:

$$\frac{\frac{1}{2}\frac{1}{2}}{\frac{1}{2} + \frac{1}{2}\frac{1}{2}}$$

Alternatively, there are 4 equally possible outcomes (2 are heads from double headed, 1 head from standard and tail from standard). There are 3 heads possible. Of those 3 heads, only one come from the standard, giving us:

$$\frac{1}{3}$$

B8. I have a fair three sided die and your discussion section has 30 people in it. If I roll a 1, I pick one person from your discussion section to answer a question in class. If I roll a 2, I pick two people from your discussion section. If I roll a 3, I pick three people from your discussion section. What is the probability that I pick you to answer a question in class?

Each possible outcome has a probability of $\frac{1}{3}$ to occur. Using the law of total probability:

$$P(you) = P(you|1)P(1) + P(you|2)P(2) + P(you|3)P(3)$$

If there's one person chosen, then you were 1 choice of 30, giving $P(you|1) = \frac{1}{30}$.

If there's two people chosen, then there are $\binom{30}{2}$ choices total. You are in 29 of those (select you, then there are $\binom{29}{1}$) people to choose from. That gives us:

$$\frac{29}{\binom{30}{2}} = \frac{29 \cdot 2! \cdot 28!}{30!} = \frac{2}{30}$$

If there are three people chosen, then there are $\binom{30}{3}$ choices total. You are in $\binom{29}{2}$ of those. That gives us:

$$\frac{\binom{29}{2}}{\binom{30}{3}} = \frac{29!3!27!}{2!27!30!} = \frac{3}{30}$$

Final answer:

$$\frac{1}{3}\frac{1}{30} + \frac{1}{3}\frac{2}{30} + \frac{1}{3}\frac{3}{30}$$

A Level Questions

A1. You go on vacation for five days, Monday through Friday. You can snorkel, tan, or go horseback riding once per day (one activity, one time per day). If you tan on Wednesday, you can't go horseback riding on Thursday. You cannot repeat events on consecutive days. How many ways can you plan your event schedule?

This is straight from homework.

Case 1: You tan on wednesday. You can't horseback ride or tan on Thursday, so you've then got one choice for Thursday too. You have 2 choices for what to do on Friday (because you can't snorkel). You have 2 choices for Tuesday (because you tanned on Wednesday), and 2 more choices for Monday (because you did something else on Tuesday). Total number of possibilities in this case: 2^3.

Case 2: You don't tan on Wednesday. You've got 2 choices for Wednesday. Because you did something on Wednesday, you have 2 options each for Tuesday and Thursday. You then also have 2 options for Monday and Friday, because you did something on Tuesday and Thursday. We start in the middle because that's what we conditioned on. Total number of possibilities in this case: 2^5.

Final answer: $2^3 + 2^5$.

A2. I select six students from a group of 40 to take a picutre with the chancellor and me. Since I have a problem with authority figures, I won't be standing next to the chancellor. two students in the class are identical twins who are identically dressed. How many possible photographs are there?

The twins make this complicated. So let's split it into 3 cases:

Case 0 twins: You choose $\binom{38}{6}$ students to take a picture of. We then need to think of how many ways we can take the picture. We've got a line of 8 people. Let's put Weisbart first, then there are 6 options for where the Chancellor could be. If Weisbart is second, then there are 5 options for the chancellor. Continue this to get: $6+5+4+3+2+1 = 21$ positions for Weisbart and the chancellor. You can then switch the order of Weisbart and the chancellor 2! ways. Lastly, those 6 students have 6! ways to order themselves in the remaining positions. Giving us:

$$\binom{38}{6} \cdot 21 \cdot 2! \cdot 6!$$

Case 1 twin: You choose $\binom{38}{5}$ from the other students. Everything else doesn't change, giving us:

$$\binom{38}{5} \cdot 21 \cdot 2! \cdot 6!$$

Case 2 twins: You choose $\binom{38}{4}$ from the other students. The students have $\frac{6!}{2!}$ ways to order themselves, giving us a total of:

$$\binom{38}{4} \cdot 21 \cdot 2! \cdot \frac{6!}{2!}$$

Final answer:

$$\binom{38}{6} \cdot 21 \cdot 2! \cdot 6! + \binom{38}{5} \cdot 21 \cdot 2! \cdot 6! + \binom{38}{4} \cdot 21 \cdot 2! \cdot \frac{6!}{2!} = \binom{39}{6} \cdot 21 \cdot 2! \cdot 6! + \binom{38}{4} \cdot 21 \cdot 2! \cdot \frac{6!}{2!}$$

The latter answer is if you did a case with ≤ 1 twin. Note that this matches John's answer, but he did it in another way.

A3. You go to the store and get either five identical flash drives or eight identical blue flash drives. When you go home, you copy four of the 20 files on your computer onto each of your flash drives. How many possible collections of flash drives can be made in this way?

Let's suppose you choose the red flash drives. You've got $\binom{20}{4}$ sets of 4 files (fingers) and 5 drives (rings) of those types to bring. That gives us:

$$\binom{\binom{20}{4} - 1 + 5}{5}$$

When you've got the blue drives, you get the same thing with different numbers:

$$\binom{\binom{20}{4} - 1 + 8}{8}$$

A4. You and your friend are in a class of 20 people. Five people will be selected from the 20 and lined up for a group picture. What is the probability that you and your friend will be selected and will stand side-by-side in the photo?

For this, we remember that $P(win) = \frac{\#wins}{\#total}$. Let's do the easy part first: $\#total$. First select the people ($\binom{20}{5}$). You then line them up, with 5! orders. That gives:

$$\#total = 5!\binom{20}{5} = \frac{20!}{15!} = P(20,5)$$

Now $\#wins$. You force you and your friend to be selected, so you have $\binom{18}{3}$ other people to choose. If you and your friend are next to each other, you count as one person. Then you have 4 people to order in 4 spots: 4!. But wait, you and your friend can switch places next to each other, giving us another 2!. That gives:

$$\#wins = \binom{18}{3}4!2!$$

Final answer:

$$\frac{\binom{18}{3}4!2!}{\frac{20!}{15!}}$$

A5. There are 60 notecards: 20 are red, 20 are green, and 20 are blue. You are handed four ntoe cards and your eyes are closed. You are told that you are holding at least 3 green cards. What is the probability that all of the cards that you are holding are green?

We know that:

$$P(4G| \geq 3G) = \frac{P(4G)}{P(\geq 3G)} = \frac{P(4G)}{P(3G) + P(4G)}$$

How many ways can we get $4G$: $\binom{20}{4}$. How many ways can we get $3G$: $\binom{20}{3}\binom{40}{1}$. The first number choses how many ways to get the greens and the last one is how to get the other cards. Final answer:

$$P(4G| \geq 3G) = \frac{\binom{20}{4}}{\binom{20}{3}\binom{40}{1} + \binom{20}{4}}$$

A6. You have three dice in a bag, a six sided die, a four sided die, and a 10 sided die. You randomly pick two dice from a bag and roll them. All dice have an equal likelihood of being chosen. What is the probability that the sum of your two rolls is 5?

First of all, this has A LOT of cases. I read this and said: that's a lot of work. You probably wouldn't choose this one unless the others confused you. (In fact, only a small amount of people in my section even attempted this problem.) This is a law of total probabilities problem:

$$P\left(\sum = 5\right) = P\left(\sum = 5|6,4\right) P(6,4) + P\left(\sum = 5|6,10\right) P(6,10) + P\left(\sum = 5|4,10\right) P(4,10)$$

All of the dice choice probabilities are the same. There are $\binom{3}{2}$ choices of die. We've listed all of them, so they are all $\frac{1}{3}$ probability.

Now we need to figure out the conditional statements.

Case 6 and 4. We have $\{4+1, 3+2, 2+3, 1+4\}$, giving us:

$$P\left(\sum = 5|6,4\right) = \frac{4}{6 \cdot 4} = \frac{4}{24}$$

Case 6 and 10. We have $\{4+1, 3+2, 2+3, 1+4\}$, giving us:

$$P\left(\sum = 5|6,4\right) = \frac{4}{6 \cdot 10} = \frac{4}{60}$$

Lastly, case 4 and 10. We have $\{4+1, 3+2, 2+3, 1+4\}$, giving us:

$$P\left(\sum = 5|6,4\right) = \frac{4}{4 \cdot 10} = \frac{4}{40}$$

Final answer:

$$\frac{1}{3}\left[\frac{4}{24} + \frac{4}{60} + \frac{4}{40}\right]$$