Math3C: Quiz 5

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Note that these answers are not reviewed by Dr. Weisbert and/or may include some errors (as we figured out first week). If you find one or more, let me know and I'll edit the document. My email is wesleytk@ucla.edu.

Suppose that a certain disease has a prevalence of $\frac{1}{5}$ in a population. You do not have the disease and can only be exposed to the disease by having physical contact with an infected person. A test for Disease D has 5% false positives and 10% false negatives. The probability that you contract the disease on being exposed is $\frac{1}{20}$.

C1 Level Problem. You randomly meet strangers until you contract Disease D. You have physical contact with each stranger one time. Once you contract Disease D, you stop meeting people. What is the probability that you contract the disease on meeting the fifth person?

This screams geometric distribution, so we need to think about what p is. This is:

$$\begin{aligned} P(\text{Contract on one contact}) =& P(\text{Person has disease})P(\text{You get the disease}|\text{Person has disease}) \\ =& \frac{1}{5}\frac{1}{20} \end{aligned}$$

Then we remember the geometric formula:

$$P(X=5) = (1-p)^4 p = \left(1 - \frac{1}{5}\frac{1}{20}\right)^4 \frac{1}{5}\frac{1}{20}$$

C2 Level Problem. You randomly meet strangers until you contract Disease D. You have physical contact with each stranger one time. What is the expected number of strangers you meet?

This is just the mean of a Geometric distribution. We know that it's:

$$E(X) = \frac{1}{p} = \frac{5 \cdot 20}{1} = 5 \cdot 20$$

B1 Level Problem. Suppose that you do not know the prevalence of a disease in a population. You randomly sample 20,000 people from the population and find that 2,000 have the disease. Find an estimate of the probability that the prevalence of the disease is between 9% and 11%?

This is a Chebychev problem, straight up, with the usual trick.

$$\begin{aligned} P(|X - E(X)| \geq c) \leq& \frac{Var(X)}{c^2} \\ P(|X - E(X)| \geq c) =& 1 - P(|X - E(X)| \leq c) \\ 1 - P(|X - E(X)| \leq c) \leq& \frac{Var(X)}{c^2} \\ P(|X - E(X)| \leq c) \leq& 1 - \frac{Var(X)}{c^2} \end{aligned}$$

Let's change 2,000 to a proportion: 10%. Then $c = 1\%$ because 9% and 11% are 1% from 10%.

$$P(|X - E(X)| \leq 0.01) \leq 1 - \frac{1}{4 \cdot 20,000 \cdot 0.01^2} = 0.875$$

B2 Level Problem. What is the probability that you contract Disease D from the second person you meet given that you do not meet more than four people?

This is conditional probability, straight up:

$$
\begin{aligned}
P(X=2|X\leq 4) =& \frac{P(X=2\cap X\leq 4)}{P(X\leq 4)} = \frac{P(X=2)}{P(X=1)+P(X=2)+P(X=3)+P(X=4)} \\
=& \frac{\left(1-\frac{1}{5}\frac{1}{20}\right)^2\frac{1}{5}\frac{1}{20}}{\left(1-\frac{1}{5}\frac{1}{20}\right)^0\frac{1}{5}\frac{1}{20}+\left(1-\frac{1}{5}\frac{1}{20}\right)^1\frac{1}{5}\frac{1}{20}+\left(1-\frac{1}{5}\frac{1}{20}\right)^2\frac{1}{5}\frac{1}{20}+\left(1-\frac{1}{5}\frac{1}{20}\right)^3\frac{1}{5}\frac{1}{20}}
\end{aligned}
$$

A1 Level Problem. Suppose you have a single partner who tests negative for Disease D. What is the probability that you contract the disease from your partner if you come into contact with your partner 10 times?

This is two step:

$$
P(D|-) = \frac{P(-|D)P(D)}{P(-|D)P(D)+P(-|D^c)P(D^c)}
$$

Now we need to figure out those probabilities:

$$
\begin{aligned}
P(D) =& \frac{1}{5} \\
P(D^c) =& \frac{4}{5} \\
P(+|D^c) =& 0.05 \\
P(-|D^c) =& 1-0.05=0.95 \\
P(-|D) =& 0.1
\end{aligned}
$$

Plugging that in:

$$
P(D|-) = \frac{P(-|D)P(D)}{P(-|D)P(D)+P(-|D^c)P(D^c)} = \frac{0.1\cdot\frac{1}{5}}{0.1\cdot\frac{1}{5}+0.95\cdot\frac{4}{5}} \approx 0.026
$$

Then we have the p for a Geometric as:

$$
P(X=1) = p \approx \frac{1}{20}\cdot 0.026
$$

BUT, when you come into contact with your friend 10 times, you're not re-testing your friend every time. Then:

$$\begin{aligned}
P(10 \geq X \geq 1|n=10) =& P(D|-)P(10 \geq X \geq 1) \\
=& P(D|-)\left[1 - P(X \geq 11)\right] \\
=& P(D|-)\left[1 - \sum_{k=11}^{\infty}(1-p)^{k-1}p\right] \\
=& P(D|-)\left[1 - (1-p)^{10}p\sum_{k=0}^{\infty}(1-p)^{k}\right] \\
=& P(D|-)\left[1 - (1-p)^{10}p\frac{1}{1-(1-p)}\right] \\
=& P(D|-)\left[1 - (1-p)^{10}p\frac{1}{p}\right] \\
=& P(D|-)\left[1 - (1-p)^{10}\right] \\
=& P(D|-)\left[1 - \left(\frac{19}{20}\right)^{10}\right] \\
\approx& 0.026 \cdot \left[1 - \left(\frac{19}{20}\right)^{10}\right]
\end{aligned}$$

Note that you don't have to do it through the infinite sums. You also can think about the probability that you don't get it on the 1st through 10th trials as 10 trials, each with a probability of $\frac{19}{20}$.

A2 Level Problem. A coin has probability $\frac{1}{7}$ of landing on heads. What is the expected number of times you must toss the coin to land on heads three times in a row?

This is a classic conditional expectation. We need to think about what needs to happen until you restart the process of tossing because you failed.

$$\begin{aligned}
E(N) =& P(HHH)E(N|HHH) + P(HHT)E(N|HHT) + P(HT)E(N|HT) + P(T)E(N|T) \\
=& \frac{1}{7^3}[3] + \frac{6}{7^3}[3+E(N)] + \frac{6}{7^2}[2+E(N)] + \frac{6}{7}[1+E(N)] \\
\left[1 - \frac{1}{7^3} - \frac{1}{7^2} - \frac{1}{7}\right]E(N) =& \frac{3 + 3\cdot 6 + 2\cdot 7\cdot 6 + 6\cdot 7^2}{7^3} \\
\frac{1}{7^3}E(N) =& \frac{3 + 18 + 84 + 6\cdot 49}{7^3} \\
E(N) =& 3 + 18 + 84 + 294 = 399
\end{aligned}$$